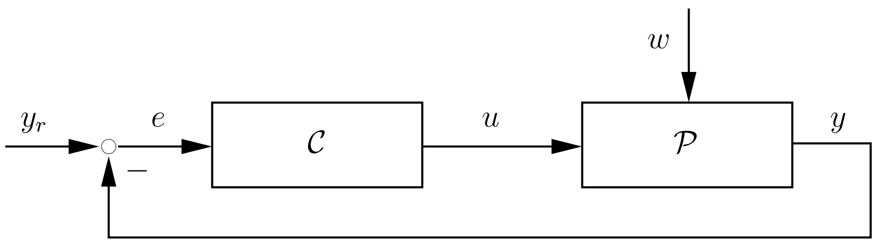
A Short Introduction to Mathematical Systems Theory and Robust Control

1 Introduction

In mathematical systems theory *output regulation* is a problem of choosing the *input* u of the system $\mathcal{P}$ such that the *output* y of this system approaches some specified *reference signal* y_r with time. In addition, it is required that this happens despite external *disturbance signals* w.



In the figure $\mathcal{C}$ is an *error feedback controller*, which produces the input u of the system $\mathcal{P}$ using the *regulation error* $y - y_r$. This kind of control structure is commonly used in practical applications such as in robotics and process engineering. Similar feedback structures can also be found in nature.

In *robust regulation* we require that the output y of the system $\mathcal{C}$ converges to the reference signal y_r even if the parameters of the plant experience small perturbations. Robustness is an essential property for the controllers used in practical applications, because the mathematical modeling of the real world system often requires making approximations and linearizations.

Also robust regulation can be found in nature. One example of this is *thermoregulation*, which keeps the core body temperature of humans and animals within certain limits regardless of the changes in external temperature and small changes within the body itself.

2 Robust Output Regulation Problem

Linear control theory considers the control of systems described by equations

$$\dot{x}(t) = Ax(t) + Bu(t) + w_s(t), \qquad x(0) = x_0 \tag{1a}$$
$$y(t) = Cx(t) + Du(t) + w_m(t) \tag{1b}$$

Here x, u and y are called the *state*, the *input* and the *output* of the system, respectively. The equation (1a) is an abstract differential equation on a Banach space X and $u(t) \in U$, $w_s(t) \in X$, $w_m(t) \in Y$ and $y(t) \in Y$ for all times $t \geq 0$, and W, U and Y are Hilbert spaces. We assume that the linear operator $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup [2] on X and the rest of the operators are bounded and linear such that $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$.

In finite-dimensional control theory the state space X is an n-dimensional linear space $\mathbb{C}^n$ and the operators A, B, C and D are matrices of appropriate dimensions. In this case equation (1a) is a system of ordinary differential equations. Infinite-dimensional state spaces allow us to consider more general systems. These include systems which are described by delay equations and partial differential equations. A more detailed introduction to the properties of infinite-dimensional systems can be found for example in [2].

The reference and disturbance signals are assumed to be generated by an *exosystem* of form

$$\begin{aligned} \dot{v}(t) &= Sv(t), \qquad v(0) = v_0 \\ y_r(t) &= F_r v(t) \\ w_s(t) &= E_s v(t) \\ w_m(t) &= E_m v(t). \end{aligned} \tag{2}$$

Here (2) is an abstract differential equation on a Hilbert space W. We assume S generates a strongly continuous group on W and that F_r, E_s and E_m are bounded linear operators.

The error feedback controller is described by equations

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_r(t)), \qquad z(0) = z_0 \tag{3a}$$
$$u(t) = Kz(t), \tag{3b}$$

where (3a) is an abstract differential equation on a Banach space Z, the linear operator $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a strongly continuous semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$.

The *output regulation problem* can now be stated as follows:

Definition 1 (Output Regulation Problem (ORP))**.** Choose controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that for all initial states $x_0 \in X$, $v_0 \in W$ and $z_0 \in Z$ the output $y(t)$ of the system (1) approaches $y_r(t)$ asymptotically, i.e.

$$\lim_{t\to\infty} \|y(t) - y_r(t)\|_Y = 0. \tag{4}$$

■

In the problem of Robust Output Regulation we attempt to choose the controller parameters in such a way that the output regulation property (4) holds even if the operators $A, B, C, D, E_s, E_m, F_r$ are replaced by operators $A', B', C', D', E'_s, E'_m, F'_r$. A more detailed formulation of the Robust Output Regulation Problem can be found for example in [3] or in the finite-dimensional case in [5, Chap. 1].

For more information on the properties of infinite-dimensional linear systems, Output Regulation and Robust Output Regulation, see for example [2, 1, 3]

3 Current Research Topics

In this section we outline some of our current research topics.

3.1 Robust Output Regulation for Infinite-Dimensional Exosystems

One of our current research topics is generalizing the state space theory of robust output regulation of infinite-dimensional systems for more general classes of disturbance and reference signals. Even when considering the control of infinite-dimensional systems, the exosystem is most commonly assumed to be finite-dimensional. These exosystems are able to generate signals which are linear combinations of signals of form

$$t \mapsto t^k e^{i\omega_n t},$$

where $k \in \mathbb{N}_0$, $\omega_n \in \mathbb{R}$. If we allow the signal generator to be infinite-dimensional we are able to study robust regulation of classes of bounded uniformly continuous signals. These include, for example, *almost periodic* signals. These are functions which can be uniformly approximated by trigonometric polynomials (i.e. functions of form $p_N(t) = \sum_{n=1}^{N} a_n e^{i\omega_n t}$). An example of an almost periodic function is

$$f(t) = \sin(t) + \sin(\sqrt{2}t).$$

This generalization gives rise to many interesting problems. For example, it is usually required that the asymptotic convergence of the output y to y_r is exponentially fast. This is always the case if the system is finite-dimensional and also possible under reasonable assumptions if the exosystem is finite-dimensional. However, if the signal generator is infinite-dimensional this can be impossible to achieve. Because of this, we need to consider *strong convergence* of the output instead of exponential convergence.

Most recent results on this topic can be found in [3, 4].

3.2 The Internal Model Principle and Infinite-Dimensional Sylvester Equations

One of the most important results in classical control theory is the *Internal Model Principle*. This result states that a feedback controller $\mathcal{C}$ solving the robust output regulation problem necessarily contains a copy of the dynamics of the exosystem producing the considered reference and disturbance signals. One of the current research topics is the generalization of the Internal Model Principle for infinite-dimensional systems and infinite-dimensional exosystems.

This problem is also closely related to the study of certain infinite-dimensional Sylvester-equations. In particular the so-called *regulator equations*

$$\Sigma S = A_e \Sigma + B_e \tag{5a}$$
$$0 = C_e \Sigma + D_e \tag{5b}$$

are important because it can be shown that the output regulation problem has a solution whenever the equations (5) have a solution Σ. Here S is the operator from the exosystem (2) and A_e, B_e, C_e and D_e are operators from the *closed-loop system* which describes the joint behaviour of the system $\mathcal{P}$ and the controller $\mathcal{C}$.

Most recent results on this topic can be found in [11, 9, 10, 8].

3.3 Infinite Zeros of Distributed Parameter Systems and The Solvability of the Output Regulation Problem

Most recent results on this topic can be found in [6, 7].

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