Robust Extended Kalman Filtering in Hybrid Positioning Applications

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Abstract—The Kalman filter and its extensions has been widely studied and applied in positioning, in part because its low computational complexity is well suited to small mobile devices. While these filters are accurate for problems with small nonlinearities and nearly gaussian noise statistics, they can perform very badly when these conditions do not prevail. In hybrid positioning, large nonlinearities can be caused by the geometry and large outliers (blunder measurements) can arise due to multipath and non line-of-sight signals. It is therefore of interest to find ways to make positioning algorithms based on Kalman-type filters more robust. In this paper two methods to robustify the Kalman filter are presented. In the first method the variances of the measurements are scaled according to weights that are calculated for each innovation, thus giving less influence to measurements that are regarded as blunder. The second method is a bayesian filter that approximates the density of the innovation with a non-gaussian density. Weighting functions and innovation densities are chosen using Hubers min-max approach for the epsilon contaminated normal neighborhood, the p-point family, and a heuristic approach. Six robust extended Kalman filters together with the classical extended Kalman filter (EKF) and the second order extended Kalman filter (EKF2) are tested in numerical simulations. The results show that the proposed methods outperform EKF and EKF2 in cases where there is blunder measurement or considerable linearization errors present.

I. INTRODUCTION

Algorithms for real time hybrid positioning need to have low computational complexity in order to be used in today’s mobile devices. Although bayesian filtering provides a theoretically complete framework for optimal nonlinear filtering, it requires the evaluation of multi-dimensional integrals which in general have to be computed using approximate numerical techniques such as Monte Carlo. High accuracy can often only be achieved with a huge amount of computation time and memory. In the linear-gaussian case the integrals can be solved analytically and the computation consists only of matrix multiplications, sums, etc. The resulting filter, i.e. the Kalman filter, has also low memory requirements, because due to the gaussianess of the distributions, only the two first moments of the distributions need to be stored. Kalman filtering has been the subject of extensive study in hybrid positioning applications in recent years. Simulations and preliminary test results with real data show that the extended Kalman filter (EKF) or the second order extended Kalman filter (EKF2) might be used in hybrid positioning applications in todays mobile devices.

The Kalman filter is based on the assumption of linear state and measurement relations and gaussian noise and state variables. The noise in the measurements is a combination of errors from many different sources and generally does not have a gaussian distribution. The performance of the filter may be seriously degraded due so-called blunder measurements. This kind of corrupted measurements may occur due the physical environment e.g. buildings, trees or some geographical obstacles. A robust filter should be able to detect the blunder measurements and to handle them accordingly.

One problem in hybrid positioning is that sometimes there is hardly enough measurements for a unique position solution and discarding a measurement completely might not be an option. The motivating question for this paper is practically what can one do with the blunder measurements besides discarding them. Another problem in hybrid positioning is that the measurement functions are non-linear functions of the state while the Kalman filter solves a linear filtering problem. Linear approximations of the Kalman filter, namely EKF and EKF2 have been used, but the estimates tend to become biased and inconsistent when the linearization errors are large [1]. GPS measurements do not have the problem with the non-linearity because of the distant location of the satellites. WLAN or mobile phone signals however have much smaller range and thus the non-linearities can have a significant effect on the estimation errors.

This paper is organized as follows. The next section consists of the formulation of the hybrid positioning problem in a bayesian framework. The measurement model and the state model are described and one approximate solution, namely the extended Kalman filter, is given. Section III starts with a brief glance at Hubers minimax ideology and two classes of densities, the $\epsilon$-contaminated neighbourhood and the p-point family, are defined. Hubers minimax approach is applied to these two classes and robust estimators are derived for use in robust filter design. A heuristic robust estimator is also proposed. Section IV is divided into two distinct parts both describing slightly different approaches for robust linear filtering. The first approach is to formulate the Kalman filter as a deterministic weighted least squares problem and to make the filter more robust by proper weighting. The second approach is an approximate bayesian filter. Kalman-type filters calculate the posterior estimates using the distribution of the innovation variable. Approximating the distribution with proper non-gaussian distribution should result in more robust filter behaviour. Altogether six robust filters are derived. The
II. Model

The state of the system at time $t_k$ ($k = 1, 2, \ldots$) is modelled as a stochastic variable $x_k$. The evolution of the state in time is given by a stochastic difference equation

$$x_{k+1} = g_k(x_k, w_k)$$

(1)

and the initial state $x_0$ is assumed to be normally distributed with mean $x_0$ and covariance matrix $P_0$. The measurements $y_k$ are related to the state by

$$y_k = h_k(x_k, v_k).$$

(2)

Here $w_k$ describes the noise in the state dynamics and $v_k$ the noise in the measurements. In this paper the noises are assumed to be white zero-mean mutually independent random processes, and also independent with the initial state. Let $y_{1:k} = \{y_i, i = 1, \ldots, k\}$ be the measurements before time step $t_k$. The measurements used in this work are GPS pseudorange measurements and their derivatives, and base station range and altitude measurements. Because there is unknown clock bias in the satellite measurements, difference measurements are used instead. One satellite is chosen as reference and all the differences are formed with respect to it. Let $r_s$ denote the position of the user, $r_s$ the position of the satellite (basetation) and $r_{u0}$ the reference satellite. The range measurements are

$$h_{k+1}(r_{k+1}) = ||r_s - r_{k+1}||$$

(3)

and the difference measurements

$$h_{k+1}(r_{k+1}) = ||r_s - r_{k+1}|| - ||r_{u0} - r_{k+1}||$$

(4)

The problem now is to solve the conditional probability density function (cpdf) of the posterior state, that is the cpdf of $x_k$ conditioned on $y_{1:k} = y_{1:k}$. Using the Bayes rule with the assumption of white noise sequences the cpdf may be formulated as

$$p(x_k|y_{1:k}) = \frac{p(x_k|y_{1:k-1})p(y_k|x_k)}{p(y_k|y_{1:k-1})}$$

(5)

where $p(x_k|y_{1:k-1})$ is the prior density, i.e. the cpdf of $x_k$ conditioned on $y_{1:k-1} = y_{1:k-1}$, $p(y_k|x_k)$ is the likelihood function giving the probability of measurement $y_k$ given state $x_k$ and $p(y_k|y_{1:k-1})$ is the predicted measurement density. To derive statistical quantities like mean and covariance from cpdf (5) requires integration of numerous multidimensional integrals using numerical integration methods. Grid based methods and sequential Monte Carlo (particle filter) methods have been used successfully to calculate the posterior estimates. They however require a lot of computation and are not suitable in real time positioning with today’s mobile devices. In the next section, one approximate analytic solution, namely the extended Kalman filter, is introduced.

A. Extended Kalman Filter

The extended Kalman filter is an approximate analytic solution to (5) if the noises $w_k$ and $v_k$ are additive gaussian noise sequences. The state and measurement functions are linearized according to

$$G_k = \frac{\partial g_k(x_k)}{\partial x_k} \bigg|_{x_k=\hat{x}_k}$$

$$H_k = \frac{\partial h_k(x_k)}{\partial x_k} \bigg|_{x_k=\hat{x}_k}$$

(6)

where $\hat{x}_k$ and $\hat{x}_k$ are the prior and posterior mean estimates at time step $t_k$. The system model becomes

$$x_{k+1} = G_k x_k + w_k$$

(7)

$$y_k = H_k x_k + v_k$$

(8)

Now the prior density is gaussian with mean and covariance given as

$$E(x_k|y_{1:k-1}) = \hat{x}_k$$

$$V(x_k|y_{1:k-1}) = \hat{P}_k = G_k \hat{P}_{k-1} G_k^T + Q_k$$

(9)

(10)

where $Q_k$ is the covariance of $w_k$. The posterior mean may be shown to be

$$E(x_k|y_{1:k}) = \hat{x}_k = \hat{x}_k + \hat{P}_k H_k^T s(y_k - H_k \hat{x}_k)$$

(11)

where $s(y_k - H_k \hat{x}_k) = -\nabla \ln p(y_k|y_{1:k-1})$. Denoting the covariance of $v_k$ with $R_k$ the posterior mean becomes

$$\hat{x}_k = \hat{x}_k + K_k (y_k - H_k \hat{x}_k)$$

(12)

and the posterior covariance is given by

$$\hat{P}_k = (I - K_k H_k) \hat{P}_k$$

(13)

where $K_k = \hat{P}_k H_k^T (H_k \hat{P}_k H_k^T + R_k)^{-1}$ is the Kalman gain matrix. $R_k$ is assumed to be positive definite to make sure that the inverse exists.

III. Classes of Densities

As seen in the previous section, Kalman filtering is based on the assumption of gaussian measurement noise and gaussian prior distribution. As argued before, the strict assumption of gaussian measurement noise may not be reasonable in hybrid positioning applications and therefore some “more robust” densities are presented here.

Huber [2] proposed a game where Nature choses the distribution $F \in F$ and the engineer chooses the estimator $T \in T$, and the asymptotic variance $V(T,F)$ is the pay-off to the engineer. The asymptotic variance of an estimator is the variance of the estimator when the sample size tends to infinity. It can be shown that for certain classes of distributions there exists a min-max solution $(T_0, F_0)$ such that

$$\min_{T \in T} \max_{F \in F} V(T,F) = V(T_0, F_0) = \max_{F \in F} \min_{T \in T} V(T,F).$$

$F_0$ is called the least favorable distribution of class $F$ and $T_0$ the min-max robust estimator which is actually the maximum likelihood estimator for the least favorable density $F_0$. 

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A. The \(\epsilon\)-contaminated normal neighborhood

Huber [2] considered the case where \(\mathcal{F}\) is the \(\epsilon\)-contaminated normal neighborhood

\[
\mathcal{F}_\epsilon = \{ F \mid F = (1-\epsilon)\phi + \epsilon H, \quad H \text{ continuous symmetrical pdf}\}, \tag{14}
\]

where \(\phi\) is the standard normal probability density and \(H\) is a continuous symmetrical probability density function. The least favorable density for the \(\epsilon\)-contaminated normal neighborhood is

\[
p^0_\epsilon(\theta) = \begin{cases} 
(1-\epsilon)e^{-\frac{1}{2}\theta^2} & \text{if } |\theta| \leq k \\
(1-\epsilon)e^{-\frac{1}{2}
k^2-\sqrt{k} \theta} & \text{if } |\theta| > k
\end{cases} \tag{15}
\]

where the threshold parameter \(k\) is given by

\[
2\phi(k) - 2\Phi(-k) = \frac{\epsilon}{1-\epsilon} \tag{16}
\]

\(\phi\) is the standard normal pdf and \(\Phi\) the standard normal cdf. The corresponding min-max robust estimator is based on the maximization of the likelihood score of the least favorable density, namely

\[
s_H(\theta) = \frac{\partial}{\partial \theta} \ln p^0_\epsilon(\theta) = \begin{cases} 
-\theta, & \text{if } |\theta| \leq k \\
-k \text{ sign}(\theta), & \text{if } |\theta| > k
\end{cases} \tag{17}
\]

Filters using this score function are labelled "H".

B. The \(p\)-point family

Martin and Masreliez [3] assert that if \(\mathcal{F}\) is the \(p\)-point family

\[
\mathcal{F}_p = \{ F \mid \int_{-y_p}^{y_p} F(\theta)d\theta = p/2 = \Phi(-y_p), \quad F \text{ symmetric and continuous at } \pm y_p\}, \tag{18}
\]

where \(\Phi\) is the standard normal cumulative distribution function, then the corresponding least favorable density is given by

\[
p^0_p(\theta) = \begin{cases} 
K \cos^2\left(\frac{\theta}{2my_p}\right), & \text{if } |\theta| \leq y_p \\
K \cos^2\left(\frac{1}{2m}\right)e^{2Kp^{-1} \cos^2\left(\frac{1}{2m}\right)(y_p-|\theta|)}, & \text{if } |\theta| > y_p
\end{cases} \tag{19}
\]

where \(K = (1-p)(y_p(1 + m \sin(\frac{1}{m})))^{-1}\) and \(m\) is given by

\[
2m - p\left[1 + \tan^2\left(\frac{1}{2m}\right)\right] - 2m + \tan\left(\frac{1}{2m}\right) = 0 \tag{20}
\]

The likelihood score of the least favorable density of the \(p\)-point family \(\mathcal{F}_p\) is

\[
s_M(\theta) = \frac{\partial}{\partial \theta} \ln p^0_p(\theta) = \begin{cases} 
-\frac{1}{my_p} \tan\left(\frac{\theta}{2my_p}\right), & \text{if } |\theta| \leq y_p \\
-\frac{1}{m} \tan(\theta) \text{ sign}(\theta), & \text{if } |\theta| > y_p
\end{cases} \tag{21}
\]

Filters using this score function are labelled "M".

C. Estimators without densities

The min-max robust estimators depend on the shape of the least favorable density. However, in more sophisticated robust filter design, it might be more appropriate to model the score function to correspond to the specific problem instead of finding the appropriate contamination classes and the least favorable densities. Andrews et. al [4] propose a three-parts-redescending estimator which is basically a heuristic modification of the estimator proposed by Huber. Because re-descending estimators reject certain measurements a modified version of the three-parts-redescending estimator is proposed here. The score function is given by

\[
s_D(\theta) = \begin{cases} 
\theta, & \text{if } |\theta| \leq k_1 \\
k_1 \text{ sign}(\theta), & k_1 < |\theta| \leq k_2 \\
k_1 k_2 / \theta, & |\theta| > k_2
\end{cases} \tag{22}
\]

where the threshold parameters \(k_1\) and \(k_2\) are chosen to fit the problem. In this work \(k_1\) is solved as \(k\) in (16) and \(k_2 = 1.5k_1\). Filters using this score function are labelled "D".

IV. ROBUST KALMAN FILTERING

In the previous section we have defined some "robust" densities and their corresponding scores. In this section we proceed to describe two different ways that these could be used to robustify the extended Kalman filter.

A. Weighted least squares filtering

The first approach is based on the robust Kalman filtering studies of Carosio et. al [5]. The Kalman filter can be shown to be equivalent to a deterministic least squares problem [6]. The filter is made more robust by replacing the least squares score function by the score functions introduced in the previous section. By approximating the derivative of the score function with a linear approximation the problem is modified to a weighted least squares problem, where the weights are calculated using the components of the innovation vector and a weighting function that corresponds to the selected score function. This approach produces filters that give less influence to measurements that are classified as blunder.

The state update equation and the measurement equation of the EKF given in (8) may be written as

\[
\begin{bmatrix} 1 \\ H_k \end{bmatrix} x_k = \begin{bmatrix} \hat{x}_k^- \\ y_k \end{bmatrix} + \begin{bmatrix} G_{k-1}(x_{k-1} - \hat{x}_{k-1}) + w_{k-1} \\ -v_k \end{bmatrix} \tag{23}
\]

To make the equations look more simple define

\[
e_k = \begin{bmatrix} G_{k-1}(x_{k-1} - \hat{x}_{k-1}) + w_{k-1} \\ -v_k \end{bmatrix} \tag{24}
\]

Now

\[
E(e_k) = 0, \quad V(e_k) = E(e_k e_k^T) = \begin{bmatrix} \hat{P}_{k-1}^- & 0 \\ 0 & R_k \end{bmatrix} \tag{25}
\]

where \(\hat{P}_{k-1}^-\) and \(R_k\) are symmetric positive definite and thus non-singular. Now define square matrix \(M_k\) such that

\[
M_k^T M_k = \begin{bmatrix} \hat{P}_{k-1}^- & 0 \\ 0 & R_k \end{bmatrix}^{-1} \tag{26}
\]
Multiplying (23) by $M_k$ and defining
\[ N_k = M_k \begin{bmatrix} 1 \\ H_k \end{bmatrix}, \quad \xi_k = M_k \epsilon_k, \quad z_k = M_k \begin{bmatrix} x_k^- \\ y_k^- \end{bmatrix}, \]
which is in the form of a standard linear least squares regression problem or equivalently
\[ \hat{x}_k = \arg \min_{x_k} \sum_{i=1}^{n} ((z_k)_i - (N_k^T)_i x_k)^2 \]
(28)
where $(N_k^T)_i$ is the $i$th row of $N_k$. Now $E(\xi_k) = 0$, $V(\xi_k) = E(\xi_k \xi_k^T) = I$, and the solution to (28) is given by
\[ \hat{x}_k = (N_k^T N_k)^{-1} N_k^T z_k = \hat{x}_k^- + K_k (y_k - H_k \hat{x}_k^-) \]
(29)
which will be the posterior mean. The posterior covariance is chosen to be
\[ \hat{P}_k = (N_k^T N_k)^{-1} = (I - K_k H_k) \hat{P}_k^- \]
(30)
where $K_k = \hat{P}_k^- H_k (H_k \hat{P}_k^- H_k^T + R_k)^{-1}$ is called the Kalman gain matrix. Equations (29) and (30) are of course similar to equations (12) and (13).

Huber [2] introduced a class of estimators, called M-estimators, that minimize other functionals than the squares. Let $\rho(\cdot)$ denote the functional which will be minimized. Now using $\rho(\cdot)$ in (28) in place of $(\cdot)^2$ gives
\[ \hat{x}_k = \arg \min_{x_k} \sum_{i=1}^{n} \rho((z_k)_i - (N_k^T)_i x_k) \]
(31)
The minimum is obtained by differentiating the functional to be minimized with respect to $(x_k)_j$, $j = 1, \ldots, n_x$ and setting these partial derivatives equal to zero. Symbolically this can be written as $n_x$ equations
\[ \sum_{i=1}^{n} (N_k)_{ij} \psi_j((z_k)_i - (N_k^T)_i \hat{x}_k) = 0, \quad j = 1, \ldots, n_x \]
(32)
where $\psi_j = (\nabla \rho)_j$ and $(N_k)_{ij}$ is the $ij$th element of matrix $N_k$. The solution of (32) is somewhat difficult to obtain at least analytically because the $\psi$-functions may be non-linear. However, (32) can be approximated with a weighted least squares problem, namely
\[ \sum_{i=1}^{n} (N_k)_{ij} \omega_{k,i} ((z_k)_i - (N_k^T)_i \hat{x}_k) = 0, \quad j = 1, \ldots, n_x \]
(33)
where the weights $\omega_{k,i}$ are given by
\[ \omega_{k,i} = \begin{cases} \frac{\phi_i((z_k)_i - (N_k^T)_i \hat{x}_k^-)}{\phi_i((z_k)_i - (N_k^T)_i \hat{x}_k^-)}, & (z_k)_i \neq (N_k^T)_i \hat{x}_k^- \\ 1, & (z_k)_i = (N_k^T)_i \hat{x}_k^- \end{cases} \]
(34)
Now the solution of (33) is given by
\[ x_k = (N_k^T \Omega_k N_k)^{-1} N_k^T \Omega_k z_k = \hat{x}_k^- + K_k (y_k - H_k \hat{x}_k^-) \]
(35)
with covariance
\[ \hat{P}_k = (1 - K_k^T H_k) \hat{P}_k^- \]
(36)
where $K_k = \hat{P}_k^* H_k^T (R_k^* + H_k \hat{P}_k^* H_k^T)^{-1}$, $\hat{P}_k^* = (\hat{P}_k^-)^{1/2} \Omega_{k-1,1} (\hat{P}_k^-)^{1/2}$, $R_k^* = R_k^{1/2} \Omega_{k-1,2} R_k^{1/2}$
\[ \Omega_k = \text{diag}((\omega_{k,1}, \ldots, \omega_{k,n_x+n_y})) = \begin{bmatrix} \Omega_{k,1} & 0 \\ 0 & \Omega_{k,2} \end{bmatrix} \]
(37)
where the weights $\Omega_{k,1} \in \mathbb{R}^{n_x \times n_x}$ and $\Omega_{k,2} \in \mathbb{R}^{n_y \times n_y}$. The motivation for dividing the weight matrix into two submatrices is that the $\Omega_{k,1}$ influences only the state update and $\Omega_{k,2}$ only the state correction with measurements. So if robustness is desired only in the measurement model one might want to define $\Omega_{k,1} = I$. Depending on the application it might be reasonable to use different $\psi$-function for different components in (32).

In this paper $\psi(\cdot) = -s(\cdot)$, where the choices for the likelihood score $s$ are given in equations (17), (21) and (22). Figure 1 shows the $\psi$-functions and the corresponding weight functions. The filters using this method are denoted with prefix "WLS".

B. Bayesian filtering assuming non-gaussian innovation sequence

The following method is based on the results of Masreliez and Martin [7]. Consider a transformed version of the measurement model in (8).
\[ T_k y_k = T_k H_k x_k + T_k v_k \]
(38)
The transformation matrix $T_k = (H_k \hat{P}_k^- H_k^T + R_k)^{-1/2}$ is chosen such that $T_k^T T_k = (H_k \hat{P}_k^- H_k^T + R_k)^{-1}$, which is easily obtained for example using the singular value decomposition. Thus the transformed innovation variable has a standard normal density distribution.

The Kalman filter calculates the posterior state estimates using the innovation variable. Under the transformed measurement model (38) the posterior mean is given by
\[ \hat{x}_k = \hat{x}_k^- + \hat{P}_k^- H_k^T T_k^T s(n_k) \]
(39)
where the score $s(n_k) = -\nabla \ln p(y_k | y_{1:k-1})|_{y_k = n_k}$ and $n_k = T_k (y_k - H_k \hat{x}_k^-)$ is the transformed innovation. Masreliez and Martin assert that if $p(y_k | y_{1:k-1})$ is the least favorable density of either $F_x$ or $F_y$, then the posterior mean is given as in (39) and the posterior covariance is given by
\[ \hat{P}_k = (1 - K_k H_k) E (p(y_k | y_{1:k-1}) \nabla s^T(y_k)|_{y_k = n_k}) \hat{P}_k^- \]
(40)
If $s = s_H$ or $s = s_D$, then
\[ E (p(y_k | y_{1:k-1}) \nabla s^T(y_k)|_{y_k = n_k}) = \int p(y_k | y_{1:k-1}) \nabla s^T(y_k)|_{y_k = n_k} dy_k = 1 - 2 \Phi(k) \]
(41)
and if $s = s_M$, then
\[ E (p(y_k | y_{1:k-1}) \nabla s^T(y_k)|_{y_k = n_k}) = (my_p)^2 (1 - p(1 + \tan^2 (\frac{1}{2m}))) \]
(42)
The filters using this method are denoted with prefix "B"
V. Testing

The robust methods were implemented in a MATLAB simulation test bench for comparison between different filters. The simulation test bench was designed to produce dynamic test data similar to what could be expected in real-world personal positioning scenario. The main difference from the real data is that in the simulation the true track and correct measurement and motion models are available. The testing process consists of first generating a true track of 100 points at one second intervals with a velocity-restricted random walk model, then generating a set of base station (BS) along the track with maximum ranges set so that one to three stations can be heard from every point on the track. A GPS constellation is then simulated with an elevation mask and shadowing profile set so that only a couple of satellites are visible at a time. Finally, noisy measurements are generated for each time step from the visible satellite ranges and delta-ranges, and base stations are left out from table III.

Several track and measurement sets were generated with different measurement sources, measurement noises and blunder measurement probabilities. The resulting test bench consists of two different sets according to two different choices for the blunder measurement probability (0.00 and 0.05). Each set has base station and satellite geometries for 9 cases representing different difficult positioning situations according to the following table

<table>
<thead>
<tr>
<th>Case</th>
<th># BS</th>
<th># GPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>Case 2</td>
<td>1.5</td>
<td>0</td>
</tr>
<tr>
<td>Case 3</td>
<td>2.5</td>
<td>0</td>
</tr>
<tr>
<td>Case 4</td>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>Case 5</td>
<td>1.5</td>
<td>2</td>
</tr>
<tr>
<td>Case 6</td>
<td>2.5</td>
<td>2</td>
</tr>
<tr>
<td>Case 7</td>
<td>0</td>
<td>2.5</td>
</tr>
<tr>
<td>Case 8</td>
<td>0</td>
<td>3.5</td>
</tr>
<tr>
<td>Case 9</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

Every case contains 100 tracks, which all have the length of 100 time steps.

The test tracks were filtered with the six robust filters of this paper, and the mean and covariance of the posterior distribution recorded at each time step. The filtered solutions were compared to the true track and some statistics derived. The filter’s ability to identify blunder measurements was also tested and reported. For comparison, the data was also processed with EKF and EKF2 [1].

As expected all the B-filters outperform EKF and EKF2 in contaminated cases. It is interesting to notice however, that when there are GPS measurements available the WLS-filters fail completely. In basestation only cases the WLS-solvers sometimes outperform even the B-filters. The WLS-filters’ failure seems to be caused by the large innovations that sometimes occur with GPS-measurements. The solvers may easily give a weight of less than 0.01 for some measurements. The filters’ ability to detect the correct corrupted measurement was also quite poor. On average the filters identify wrong measurement more often than the right measurement, and this becomes a serious problem with the WLS-filters, because correct measurements get scaled close to zero. All that can be said about the blunder detection is, that it is extremely hard to tell if a measurement is a blunder measurement by measuring the magnitude of the corresponding innovation - the prior might also be wrong, but the filter identifies the measurements as blunder. In tables III and IV are listed how well the filters detect and identify blunder measurements. The actions of the filter may be divided into five categories:

I Blunder measurements are present and the filter detects them correctly. Filter calculates robust estimates.
II Blunder measurements are present but the filter identifies wrong measurement.
III Blunder measurements are present but the filter does not notice it, thus working as EKF.
IV False alarm. No blunder measurements, but filter calculates robust estimates.
V No blunder measurements and no action. This means that the filters, except the filters based on the $p$-point score, work as EKF.

The frequencies of categories 1-4 are listed in the table and the categories are denoted by the corresponding Roman numeral. The frequency of category 5 is the frequency of the complement of the union of the three first categories. The False alarms occur quite often, but they are not very dangerous except for WLS-filters with GPS-measurements. If there were blunder measurements present, the filters detected them almost every time. The most interesting thing however is the huge frequency of category 2 in the contaminated case. Because the false alarm is the only one from the categories 1-4 which can happen in the uncontaminated case, the zero frequencies are left out from table III.

VI. Conclusion

Based on the simulations, the proposed methods seem to outperform EKF and EKF2 in contaminated cases and do almost as well in normal cases. Therefore, the proposed methods should be taken into consideration to be used in mobile positioning devices. The approximate bayesian filter is to be preferred over the WLS-filter if there are GPS-measurements available. WLS-filters might be considered otherwise. The score functions should be chosen based on empirical measurement data, but without any preliminary knowledge of the situation it seems to be safe to use any of the scores proposed here. However, the modified Hampel estimator seems to give
best estimates. More sophisticated robust filter should be able to identify certain situations and apply the most convenient robust model to them, but this is beyond the scope of this paper and left for future study.

ACKNOWLEDGMENT

This study was funded by Nokia Corporation. The EKF and EKF2 used for reference were implemented by Simo Ali-Löytty and the simulation test bench was generated using Niilo Sirola’s test bench generator [8].

REFERENCES

Fig. 1. The $\psi$- and the $\omega$-functions
### TABLE I

RESULTS OF THE TESTBENCH WITH BLUNDER PROB. \( b = 0.00 \)

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### TABLE II

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