 CONSISTENCY OF KALMAN FILTER EXTENSIONS

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Aim: Determine user position and velocity using satellites and cellular base stations. User state $x$ and measurements $y$ modelled as stochastic processes.

Problem statement

Initial state $x_0 \sim N_n(x_0, P_0)$
State dynamic $x_{k+1} = f(x_k, u_k) + w_k$
$w_k \sim N_n(0, Q_k)$
Measurement equation $y_k = h_i(x_k) + v_k$
$v_k \sim N_n(0, R_k)$

Filters approximate the state conditional probability density given past and current measurements. Kalman filter solves this exactly when measurements are linear, $h_i(x_k) = h_i(x_k)$.

Kalman Filter (KF)

Prior mean $\hat{x}_0 = \mu_0$
Prior cov. $P_0 = \Sigma_0$
Posterior mean $\hat{x}_k = \hat{x}_k + K_k(y_k - H_k \hat{x}_k)$
Posterior cov. $P_k = (I - K_k H_k) P_k$
Kalman gain $K_k = P_k H_k^T (H_k P_k H_k^T + R_k)^{-1}$

We consider three Kalman filter extensions:

**EKF (Extended Kalman Filter)** Linearizes the measurement functions. Algorithm almost same as KF with $H_k$ derivative of $h_i(x_k)$.

**EKF2 (Second Order Extended Kalman Filter)** An elaboration of EKF. Takes into consideration the nonlinearity of the measurement models.

**PKF (Position Kalman Filter)** Filters a sequence of static position and velocity solutions using Kalman Filter.

Measurements

**Base station range measurements**

$$r_{i,k} = \sqrt{(x_k - r_{i})^2} + \epsilon_{r_{i,k}}$$
where $r_{i}$ is $i$th base station position, $r_{i}$ is the user position, and $\epsilon_{r_{i,k}}$ is error term.

**Satellite pseudo range measurements**

$$\rho_{i,k} = \sqrt{(v_{i,k} - v_{i})^2} + b + \epsilon_{\rho_{i,k}}$$
where $r_{i,k}$ is $i$th satellite position, $b$ is clock bias in meters and $\epsilon_{\rho_{i,k}}$ is error term.

**Satellite data range measurements**

$$\rho_{i,k} = \sqrt{(v_{i,k} - v_{i})^2} + b + \epsilon_{\rho_{i,k}}$$
where $v_{i,k}$ is $i$th satellite velocity, $v_{i}$ is user velocity, $\epsilon_{\rho_{i,k}}$ is error term.

User clock is not modelled, therefore we use the difference measurements of satellites. We assume that errors $\epsilon_{r_{i,k}}, \epsilon_{\rho_{i,k}}$, and $\epsilon_{\rho_{i,k}}$ are zero mean, independent Gaussian white noise, with $\sigma_{\rho_{i,k}} = 10 m$, $\sigma_{\rho_{i,k}} = 0.1 m/s$ and $\sigma_{\rho_{i,k}} = 80 m$.

Example of inconsistency

Figure 1: EKF veers away from the true route and gets stuck in an incorrect solution branch. EKF2 increases covariance matrix. Covariance ellipses (s.t. $(x - \hat{x})^T P_{k+1} (x - \hat{x}) = 2.2173$) are shown at time instants $t_0, t_60, t_{110}, t_{111}$, and $t_{120}$.

A filter is consistent if predicted errors are at least as large as actual errors. We use a new inconsistency test, which does not assume that distributions are Gaussian.

Figure 2: Inconsistency test of the situation of Figure 1. In this case, only EKF fails the test (test statistic larger than red line). Black dashed line is the Normalized Estimation Error Squared (NEES) test. Risk level of both test is $\alpha = 0.01$.

Simulation results

Table 1: 2D error limits for filters, based on simulation of one hundred true routes with every route run 10 times. The 95% column gives the radius that contains 95% of all 2D errors. Red numbers mark where E-112 and E-911 accuracy requirements are met.

<table>
<thead>
<tr>
<th>Meas.</th>
<th>WLS</th>
<th>PKF</th>
<th>EKF</th>
<th>EKF2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1</td>
<td>100</td>
<td>2448</td>
<td>2731</td>
<td>2160</td>
</tr>
<tr>
<td>0 2</td>
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<tr>
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<td>7</td>
<td>117</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>0 4</td>
<td>100</td>
<td>2448</td>
<td>2139</td>
<td>2119</td>
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<td>2117</td>
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<td>190</td>
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<td>187</td>
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<td>69</td>
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<tr>
<td>1 3</td>
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<td>26</td>
</tr>
<tr>
<td>1 4</td>
<td>1</td>
<td>65</td>
<td>30</td>
<td>25</td>
</tr>
</tbody>
</table>

From Table 1, note that

- EKF and EKF2 differ noticeably only with one or two base station range measurements
- PKF differs noticeably from the two others only when there is no unique position solution
- Filtering gives much better estimates than static WLS solutions

Table 2: Percentage of times when filters are inconsistent ($\alpha = 0.01$) w.r.t. the general inconsistency test (only non-zero results tabulated).

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<tr>
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<th>EKF</th>
<th>EKF2</th>
</tr>
</thead>
<tbody>
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<td>9</td>
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<tr>
<td>0 2</td>
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<tr>
<td>0 3</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

From Table 2 can be concluded that

- PKF is generally consistent
- EKF and EKF2 are inconsistent only with underdetermined systems
- Significant inconsistency when there are only base station measurements, dramatic when using only one or two base station measurements. In these cases EKF2 works better than EKF.