ON THE METHOD OF ADDING-REMOVING KNOTS FOR SOLVING THE SMOOTHING PROBLEM WITH OBSTACLES

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Let $x_i, \ i = 0, \ldots, N$, be given points on the interval $[a, b]$ and $f_i, \ i = 0, \ldots, N$, measured values at knots $x_i$.

In particular case we have $N = 22$, $x_i = 1 + \frac{i}{2}$ and $f_i$ presented on the figure.
A cubic spline interpolant $f$, satisfying conditions

$$f(x_i) = f_i, \ i = 0, \ldots, 22,$$

$$f''(1) = f''(12) = 0$$
For given weights $p_0 > 0, \ldots, p_N > 0, p > 0$ we consider a minimization problem

$$\min_{f \in H^2(a,b)} \left( p \int_a^b |f''(x)|^2 \, dx + \sum_{i=0}^N p_i (f(x_i) - f_i)^2 \right),$$

where $H^2(a,b)$ is the Sobolev space.

The solution in the case of equal weights $p_0 = \ldots = p_{22} = p = 1$
The solution in the case of non-equal weights

\[ p_{10} = p_{12} = p_{14} = 10 \]
The solution in the case of non-equal weights

\[ p_{10} = p_{12} = p_{14} = 100 \]
Let us consider a smoothing problem with obstacles as follows

\[
\min_{f \in H^2(a, b)} \int_a^b |f''(x)|^2 \, dx.
\]

\[
|f(x_i) - f_i| \leq \varepsilon_i, \quad i = 0, \ldots, N
\]
For given natural numbers $r$ and $n$, $2r > n \geq 1$, we denote

$$L_2^{(r)}(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R} \mid D^\alpha f \in L_2(\mathbb{R}^n), |\alpha| = r \}$$

as the Beppo-Levi space, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

We define an operator

$$T : L_2^{(r)}(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \times \ldots \times L_2(\mathbb{R}^n),$$

$$T f = \left\{ \sqrt{\frac{r!}{\alpha!}} D^\alpha f \mid |\alpha| = r \right\},$$

where $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$. We also need the semi-inner product

$$\langle f, g \rangle_{L_2^{(r)}(\mathbb{R}^n)} = \langle Tf, Tg \rangle = \sum_{|\alpha| = r} \int_{\mathbb{R}^n} \frac{r!}{\alpha!} D^\alpha f D^\alpha g \, dX$$

and the corresponding seminorm

$$\|Tf\| = \sqrt{\langle Tf, Tf \rangle}.$$
Let $I, I_0 \subset I$ and $I_1 = I \setminus I_0$ be finite sets of indexes. For given data $X_i \in \mathbb{R}^n$, $i \in I$, $z_i \in \mathbb{R}$, $i \in I_0$, and $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i < \beta_i$, $i \in I_1$, we consider a smoothing problem with obstacles as the problem

$$
\min_{f \in \Omega_{\alpha\beta}} \sum_{|\alpha| = r} \int_{\mathbb{R}^n} \frac{r!}{\alpha!} (D^\alpha f)^2 \, dX,
$$

(1)

where

$$
\Omega_{\alpha\beta} = \{ f \in L^2(r)(\mathbb{R}^n) \mid f(X_i) = z_i, i \in I_0, \\
\quad \alpha_i \leq f(X_i) \leq \beta_i, i \in I_1 \}.
$$

The solution of this problem is a natural spline.
Let $P_{r-1}$ be the space of polynomials of order $\leq r - 1$ on $\mathbb{R}^n$. A function

$$S(X) = Q_0(X) + \sum_{i \in I} d_i G(X - X_i), \quad X \in \mathbb{R}^n,$$

with $X_i \in \mathbb{R}^n$, $Q_0 \in P_{r-1}$ and

$$\sum_{i \in I} d_i Q(X_i) = 0 \quad \forall Q \in P_{r-1},$$

is called the natural spline.

Function $G$ is the fundamental solution of the operator $\Delta^r$, where $\Delta$ is the $n$-dimensional Laplace operator. For $n$ odd

$$G(X) = c_{rn} \|X\|^{2r-n}$$

and for $n$ even

$$G(X) = c_{rn} \|X\|^{2r-n} \ln \|X\|$$

where $c_{rn} > 0$ are some constants and $\|X\| = (x_1^2 + \ldots + x_n^2)^{1/2}$.

It is known that any natural spline belongs to $L^2_r(\mathbb{R}^n)$. 
A natural spline $S \in \Omega_{\alpha\beta}$ is the solution of the problem (1) if and only if in the knots of $I_1$ the spline $S$ satisfies the conditions

$$d_i = 0, \text{ if } \alpha_i < S(X_i) < \beta_i,$$
$$(-1)^r d_i \geq 0, \text{ if } S(X_i) = \alpha_i,$$
$$(-1)^r d_i \leq 0, \text{ if } S(X_i) = \beta_i. \tag{2}$$

According to (2) the solution $S \in \Omega_{\alpha\beta}$ is in form

$$S(X) = Q_0(X) + \sum_{i \in I_0 \cup M^\alpha \cup M^\beta} d_i G(X - X_i),$$

satisfying conditions

$$S(X_i) = z_i, \quad i \in I_0,$$
$$S(X_i) = \alpha_i, \quad i \in M^\alpha \subset I_1,$$
$$S(X_i) = \beta_i, \quad i \in M^\beta \subset I_1,$$

with $(-1)^r d_i > 0$ for $i \in M^\alpha$ and $(-1)^r d_i < 0$ for $i \in M^\beta$. 
The method of adding-removing knots consists of three steps.

At first we construct natural spline satisfying conditions $S_0(X_i) = z_i$, $i \in I_0$. If $I_0 = \emptyset$, we take $S_0 \in \mathcal{P}_{r-1}$ arbitrarily.

At second step we add all knots in which the spline $S_0$ does not satisfy the conditions of obstacles. If (2) is satisfied we have the solution of initial problem. In opposite case we go to the step 3.

At third step we repeatedly remove knots not satisfying conditions (2) until we get spline $S_1$ satisfying (2) and hence optimal on his own set of knots.

We continue the second step with $S_1$ if the conditions of obstacles are not satisfied. As result, we get splines $S_0, S_1, S_2, \ldots$, optimal on their own sets of knots.
Examples

Let us consider one-dimensional case \((n = 1)\) and \(r = 1\), then corresponding natural splines are linear splines

\[
S(x) = c + \sum_{i=0}^{N} d_i (x - x_i)_+, \\
\]

with knots \(x_0, \ldots, x_N\) where

\[
x_+ = \begin{cases} 
  x, & \text{if } x \geq 0, \\
  0, & \text{if } x < 0.
\end{cases}
\]

There is an easy way to determine the signs of coefficients \(d_i, i = 1, \ldots, N\).
Let $I_0 = \{1, 2\}$ and $I_1 = \{3, 4, 5, 6\}$. The conditions of interpolation and obstacles are given in figure
Ignatov, M. I. and Pevnyj, A. B.
Natural Splines of Several Variables.

The method is finite, i.e. $\exists k$ such that $S_k$ is the solution of initial problem.

Proof is based on a false lemma.

In the case of cubic splines $(n = 1, r = 2)$ we have an example of cycling.