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A Hyperbolic Interpretation of Cauchy Type Kernels in Hyperbolic Function Theory
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Abstract
In this paper we study a mean-value property for solutions of the Laplace-Beltrami equation

$$x_n^2 \Delta h - (n - 1)x_n \frac{\partial h}{\partial x_n} = 0$$

with respect to the volume and the surface integral on the Poincaré upper-half space $\mathbb{R}^{n+1}_{+} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_n > 0\}$ with the Riemannian metric $g = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}$. Also we represent the Cauchy type kernels in terms of the hyperbolic geometry.

1 Clifford Numbers

The Clifford algebra $\mathcal{C}_{0,n}$ is the $2^n$-dimensional real associative algebra generated by the symbols $\{e_1, ..., e_n\}$ with the multiplication rule

$$e_ie_j + e_je_i = -2\delta_{ij}.$$ 

A canonical basis of $\mathcal{C}_{0,n}$ is given by $e_A = e_{a_1} \cdots e_{a_k}$, $A = \{a_1, ..., a_k\} \subset M = \{1, ..., n\}$ and $1 \leq a_1 < \cdots < a_k \leq n$. Especially $e_{\emptyset} = 1$ and $e_{\{ij\}} = e_j$. The pseudoscalar is the element $e_M = e_1 \cdots e_n$. 

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The space of \( k \)-vectors is defined by \( \mathcal{C}_k \ell_{0,n} = \text{span}\{e_A : |A| = k\} \) and any \( a \in \mathcal{C}_n \) admits the following multivector decomposition

\[
 a = [a]_0 + [a]_1 + \cdots + [a]_n
\]

with \([a]_k \in \mathcal{C}_k \ell_{n}\). There exists the canonical embedding \( i : \mathbb{R}^{n+1} \hookrightarrow \mathcal{C}_{0,n} \) defined by

\[
 i : (x_0, ..., x_n) \mapsto x_0 + \sum_{j=1}^{n} e_j x_j.
\]

Elements of the set \( \mathbb{R}^{n+1} \cong i(\mathbb{R}^{n+1}) = \mathcal{C}_{0,n}^0 \oplus \mathcal{C}_{0,n}^1 \) are called paravectors. We abbreviate briefly \( e_0 = 1 \).

Let us also define a few involutions. Let \( \{e_1, ..., e_n\} \) an orthonormal basis for \( \mathbb{R}^n \) and \( a, b \in \mathcal{C}_n \). The main–involition is a map \( ' : \mathcal{C}_n \to \mathcal{C}_{0,n} \) defined by the relations \( e_i' = -e_i \) and \( (ab)' = a'b' \). The conjugation is an anti-involition \( \overline{\cdot} : \mathcal{C}_{0,n} \to \mathcal{C}_n \) defined by the relations \( \overline{e_i} = -e_i \) and \( \overline{ab} = \overline{b}\overline{a} \).

2 On the Poincaré Upper-Half Space

In this section we will consider the certain Riemannian manifold, called the Poincaré Upper-Half Space. Especially we are interested in computing the hyperbolic distances in practice.

In the next section the hyperbolic function theory is related to the Poincaré upper-half model \((\mathbb{R}_+^{n+1}, g)\), where the hyperbolic metric in canonical coordinates is defined by

\[
 g = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}.
\]

In general, any oriented smooth Riemannian manifold with the metric

\[
 g = \sum_{i,j=0}^{n} g_{ij} dx_i dx_j
\]

admits the coordinate expression (see [14]):

\[
 dV_g(x) = \sqrt{\det(g_{ij}) dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n}.
\]

In the canonical coordinates on the upper-half space model \( \det(g_{ij}) = 1/x_n^{2(n+1)} \). Then the volume element

\[
 dx_k := dV_g(x) = \frac{dx}{x_n^{n+1}}.
\]
where $dx = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n$ is the Euclidean volume element. We define the hyperbolic surface element on a smooth manifold-with-boundary $U$ in $\mathbb{R}^{n+1}_+$ with the codimension 0 by

$$d\sigma_h = \nu dS / x_n,$$

where $\nu$ is the unit normal field on $U$ and $dS$ the classical scalar surface element.

The metric $g$ allows us to define distances on $\mathbb{R}^{n+1}_+$. The geodesics are described more detailed in the following theorem.

**Theorem 2.1** On the Poincaré half-space model $\mathbb{R}^{n+1}_+$ geodesics are circles or lines which meet the boundary orthogonally.

**Proof.** See [20] p. 71 or [15] p. 38. ■ In the essential role in our studies are balls. Next we study balls more detailed. First we shall consider distances on the hyperbolic upper-half space. The hyperbolic upper-half space is an immersed submanifold of $\mathbb{R}^{n+1}_+$. Its tangent space at any point $x \in \mathbb{R}^{n+1}_+$ can nonetheless be viewed as a subspace of $T_x \mathbb{R}^{n+1}_+$. In addition, by dimensional reasons $T_x \mathbb{R}^{n+1}_+ = T_x \mathbb{R}^{n+1} + 1$ for each $x \in \mathbb{R}^{n+1}_+$. Let $\iota : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$ be the canonical immersion. Then we may identify $\iota(\mathbb{R}^{n+1}_+)$ and $\mathbb{R}^{n+1}_+$ as sets. That identification allow us to use two different geometric structures on $\mathbb{R}^{n+1}_+$ parallel: the hyperbolic and the Euclidean structures. Thus we may loosely speak of Euclidean distances and balls on $\mathbb{R}^{n+1}_+$, that is, we may compute the Euclidean distance $|x - y|$ for $x, y \in \mathbb{R}^{n+1}_+$.

**Lemma 2.2** The hyperbolic distance $d_h(x, a)$ between the points $x = x_0 + e_1 x_2 + \cdots + x_n e_n$ and $a = a_0 + e_1 a_1 + \cdots + a_n e_n$ in $\mathbb{R}^{n+1}_+$ is

$$d_h(x, a) = \text{arcosh} \, \lambda(x, a),$$

where

$$\lambda(x, a) = \lambda(a, x) = \frac{|x - a|^2 + 2a_n x_n}{2x_n a_n} = \frac{|x - a|^2}{2x_n a_n} + 1.$$

**Proof.** See details, e.g., [16]. ■ Next we briefly review the connection between the hyperbolic and the Euclidean distance of two points. As a direct computation we obtain the following formulae.
Lemma 2.3 If \( x = x_0 + e_1 x_2 + \cdots + x_n e_n \) and \( a = a_0 + e_1 a_1 + \cdots + a_n e_n \) are points in \( \mathbb{R}_+^{n+1} \), then
\[
|\lambda(x,a)| = \begin{cases} 
2x_n a_n (\lambda(x,a) - 1), & n \leq 2 \\
2x_n a_n (\lambda(x,a) + 1), & n \geq 2
\end{cases}
\]
where \( \lambda(x,a) = \frac{\lambda(x,a) - 1}{\lambda(x,a) + 1} = \tanh^2 \left( \frac{d_h(x,a)}{2} \right) \).

Next we shall prove the connection between the hyperbolic and the Euclidean ball in \( \mathbb{R}_+^{n+1} \).

Proposition 2.4 Let \( \iota : \mathbb{R}_+^{n+1} \to \mathbb{R}_+^{n+1} \) be the canonical immersion. Then
\[
\iota(B_h(a, R_h)) = B_e(\tau(a, R_h), R_e(a, R_h)),
\]
where
\[
\tau(a, R_h) = a_0 + e_1 a_1 + \cdots + a_{n-1} e_{n-1} + a_n e_n \cosh R_h
\]
is the Euclidean center and
\[
R_e(a, R_h) = a_n \sinh R_h
\]
is the corresponding Euclidean radius.

Proof. Let \( x \in B_h(a, R_h) \), that is \( d_h(x,a) < R_h \). This is equivalent to \( \lambda(x,a) < \cosh R_h \). Using the above lemma we may write the inequality as
\[
\frac{|x - a|^2}{2x_n a_n} - 1 < \cosh R_h.
\]
This is equivalent to
\[
|x - a|^2 < 2x_n a_n (\cosh R_h - 1).
\]
Since \( |x - a|^2 = |P(x - a)|^2 + (x_n - a_n)^2 \) we obtain, that this is equivalent to
\[
|P(x - a)|^2 + x_n^2 - 2x_n a_n \cosh R_h + a_n^2 < 0.
\]
Since \( x_n^2 - 2x_n a_n \cosh R_h = (x_n - a_n \cosh R_h)^2 - a_n^2 \cosh^2 R_h \) we obtain
\[
|P(x - a)|^2 + (x_n - a_n \cosh R_h)^2 < a_n (\cosh^2 R_h - 1) = a_n \sinh^2 R_h.
\]
Thus we see that if we define $\tau(a, R_h) = Pa + a_n e_n \cosh R_h$ and $R_e(a, R_h) = a_n \sinh^2 R_h$ we obtain

$$|x - \tau(a, R_h)| < R_e(a, R_h).$$

The proof is complete. □

The preceding proposition allow us to abbreviate briefly by

$$B_h(a, R_h) = B_e(\tau(a, R_h), R_e(a, R_h)).$$

Proposition 2.4 says that if $x$ is a boundary point of the hyperbolic ball $B_h(a, R_h)$, that is $d_h(a, x) = R_h$ then the Euclidean distance between the points $x$ and $\tau(a, R_h)$ is $|x - \tau(a, R_h)| = a_n \sinh R_h$. Putting these immediate consequences together we obtain the following corollary, which has an important role in the forthcoming sections.

**Corollary 2.5** If $x \in \mathbb{R}_n^{n+1}$ and $\tau(a, x) = a_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n e_n \cosh d_h(a, x)$ then

$$|x - \tau(a, x)| = a_n \sinh d_h(x, a).$$

### 3 On Hyperbolic Function Theory

We briefly recall the definition of $k$–hypermonogenic functions. Let $\Omega$ be an open subset of $\mathbb{R}_n^{n+1}$. We consider functions $f : \Omega \to \mathcal{C}_{0,n}$, whose components are continuously differentiable. Before the definition we should define the following technical tool. We assume that the Clifford algebra $\mathcal{C}_{0,n-1}$ is generated by $\{e_1, \ldots, e_{n-1}\}$. Then for each $a \in \mathcal{C}_n$ there exist Clifford numbers $b$ and $c$ in $\mathcal{C}_{0,n-1}$ satisfying

$$a = b + ce_n.$$ 

We abbreviate briefly $Pa = b$ and $Qa = c$. The operator $P$ is obviously a projection but $Q$ is not (since $Q^2 = 0$). The decomposition

$$a = Pa + (Qa)e_n$$

is (still) called the projector decomposition. Using the projector decomposition we may define an involution $\sim : \mathcal{C}_{0,n} \to \mathcal{C}_n$ by

$$\hat{a} = Pa - (Qa)e_n.$$ 

The above involution is called the hat-involution. It is straightforward to see that if $a, b \in \mathcal{C}_n$ then $\hat{a} \hat{b} = \hat{ab}$ and if $\{e_1, \ldots, e_n\}$ is the set of generators of
\( C_{\ell_0, n} \) then \( \hat{e}_j = (-1)^{\delta_{ij}} e_j \).

Obviously the \( P \)- (resp. the \( Q \)-) part is a generalization of the real- (resp. the imaginary-) part of a complex number.

The left Dirac operator in \( C_{\ell_0, n} \) is defined by

\[
D_{\ell} f = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}
\]

and the right Dirac operator by

\[
D_r f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} e_i.
\]

The operators \( \overline{D}_\ell \) and \( \overline{D}_r \) are defined by

\[
\overline{D}_\ell f = \sum_{i=0}^{n} \overline{e}_i \frac{\partial f}{\partial x_i}, \quad \overline{D}_r f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \overline{e}_i.
\]

Let \( \Omega \) be an open subset of \( \mathbb{R}_{+}^{n+1} \). The modified Dirac operators \( M_{\ell}^k \), \( \overline{M}_{\ell}^k \), \( M_{r}^k \) and \( \overline{M}_{r}^k \) are introduced in [2] and [7] by

\[
M_{\ell}^k f (x) = D_{\ell} f (x) + k \frac{Q' f}{x_n},
\]

\[
M_{r}^k f (x) = D_{r} f (x) + k \frac{Q f}{x_n},
\]

and

\[
\overline{M}_{\ell}^k f (x) = \overline{D}_{\ell} f (x) - k \frac{Q' f}{x_n},
\]

\[
\overline{M}_{r}^k f (x) = \overline{D}_{r} f (x) - k \frac{Q f}{x_n},
\]

where \( f \) is a continuously differentiable function on \( \Omega \) and

\[
Q' f = (Q f)', \quad P' f = (P f)'.
\]

The operator \( M_{n-1}^\ell \) is also denoted briefly by \( M \).

\[\text{1} \text{The symbol } \delta_{ij} \text{ is the well-known Kronecker's delta-function.}\]
Definition 3.1 Let $\Omega \subset \mathbb{R}^{n+1}_+$ be open. A continuously differentiable function $f : \Omega \to \mathcal{C}_{0,n}$ is left $k$-hypermonogenic, if

$$M_k^l f (x) = 0$$

for any $x \in \Omega$. The right $k$-hypermonogenic functions are defined similarly. The $(n-1)$-left hypermonogenic functions are called hypermonogenic functions.

Paravector-valued hypermonogenic functions are $H$-solutions introduced by Heinz Leutwiler in [17] and [18]. Clifford algebra-valued hypermonogenic functions are introduced by the first author and Leutwiler and in [7]. Their theory is further studied in [1], [2], [4], [5], [6], [8], [9], [10], [11], [12] and ?. We state some main properties of $k$-hypermonogenic functions.

Consider the Riemannian manifold $(\mathbb{R}^{n+1}_+, g)$, where

$$g = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$  

One may prove that the corresponding Laplace-Beltrami (cf. [?]) operator is

$$\Delta_{lb} f := x_n^2 \Delta f - (n-1)x_n \frac{\partial f}{\partial x_n}.$$  

It is an important fact that the $P$-part of a hypermonogenic function is a null-solution of the Laplace-Beltrami operator and the $Q$-part is an eigenfunction as follows.

Lemma 3.2 ([7]) Let $f : \Omega \to \mathcal{C}_n$ be twice continuously differentiable. Then

$$P(MM^f) = \Delta f - \frac{n-1}{x_n} \frac{\partial f}{\partial x_n}$$

and

$$Q(MM^f) = \Delta Q f - \frac{n-1}{x_n} \frac{\partial Q f}{\partial x_n} + (n-1) \frac{Q f}{x_n^2}.$$  

We recall the Cauchy formulae for their $P$- and $Q$-part separately. We need the notion of the manifold-with-boundary, see [19]. The existence of the outer unit normal, see [14].

Proposition 3.3 ([2]) If $f$ is a hypermonogenic function on $\Omega$ and $K \subset \Omega$ is an oriented $(n+1)$-dimensional manifold-with-boundary. Then for each $a \in K$ we have

$$P f(a) = 2^n a^n \frac{\omega_n}{\omega_{n+1}} \int_{\partial K} P(p(x,a) \nu(x) f(x)) \frac{dS(x)}{x_n^{n-1}}$$

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where \( dS \) is the scalar surface element, \( \nu \) is the outer unit normal vector field, and

\[
p(x, a) = -\frac{1}{2^{2n-1}a_n} \frac{D_t}{|x - a|^{n-1}|x - \hat{a}|^{n-1}} \int_{[x-a]} (1 - s)^{n-1} ds
\]

\[
= \frac{x_n^{n-1}(x - a)^{-1} - (x - \hat{a})^{-1}}{2a_n |x - a|^{n-1}|x - \hat{a}|^{n-1}}.
\]

**Proposition 3.4 ([2])** If \( f \) is a hypermonogenic function on \( \Omega \) and \( K \subset \Omega \) is an oriented \((n + 1)\)-dimensional manifold-with-boundary. Then for each \( a \in K \) we have

\[
Qf(a) = \frac{2^n a_n^{n-1}}{\omega_{n+1}} \int_{\partial K} Q(q(x, a)\nu(x)f(x))dS(x)
\]

where \( dS \) is the scalar surface element, \( \nu \) is the outer unit normal vector field, and

\[
q(x, a) = -\frac{1}{2(n-1)} \frac{D_t}{|x - a|^{n-1}|x - \hat{a}|^{n-1}} \frac{1}{(x - a)^{-1} + (x - \hat{a})^{-1}}
\]

\[
= \frac{1}{2} \frac{(x - a)^{-1} + (x - \hat{a})^{-1}}{|x - a|^{n-1}|x - \hat{a}|^{n-1}}.
\]

### 4 Hyperbolic Interpretation of the \( P \)- and \( Q \)-kernel

In this section we will study how we can express the \( P \)- and \( Q \)-kernels using hyperbolic geometry. First we review some necessary tools.

**Lemma 4.1** Let \( \Omega \) be an open subset of \( \mathbb{R}^{n+1} \). If \( h : \Omega \to \mathbb{R} \) and \( g : (a, b) \to \mathbb{R} \) such that \( h(\Omega) \subset (a, b) \) and are differentiable, then

\[
D_t (g \circ h)(x) = g'(h(x)) D_t h(x).
\]

and similarly

\[
\overline{D_t} (g \circ h)(x) = g'(h(x)) \overline{D_t} h(x)
\]

for any \( x \in \Omega \).

**Lemma 4.2** Let \( x \) and \( a \) be point in \( \mathbb{R}^{n+1} \). Then

\[
\frac{\partial \lambda(x, a)}{\partial x_i} = \frac{x_i - a_i - a_n (\lambda(x, a) - 1) \delta_{in}}{x_n a_n}.
\]
Proof. We calculate as usual

\[ \frac{\partial \lambda(x, a)}{\partial x_i} = \frac{x_i - a_i}{x_n a_n} - \frac{|x - a|^2}{2x_n^2 a_n} \delta_{in}. \]

Using Lemma 2.3 we infer

\[ \frac{\partial \lambda(x, a)}{\partial x_i} = \frac{x_i - a_i}{x_n a_n} - \frac{2x_n a_n (\lambda(x, a) - 1) \delta_{in}}{2x_n^2 a_n} \]
\[ = \frac{x_i - a_i - a_n (\lambda(x, a) - 1) \delta_{in}}{x_n a_n}, \]

completing the proof. ■

**Lemma 4.3** If \( a \in \mathbb{R}_+^{n+1} \) and \( \tau(a, R_h) = a_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n \cosh R_h e_n \)

then

\[ D_{\ell} \lambda(x, a) = \frac{P(x - a) - (x_n - a_n \lambda(x, a)) e_n}{x_n a_n} = \frac{x - \tau(a, R_h)}{x_n a_n} \]

and

\[ D_{\ell} \frac{|x - a|}{|x - \hat{a}|} = \frac{P(x - a) - (x_n - a_n \lambda(x, a)) e_n}{(\lambda(x, a) - 1)^{\frac{1}{2}} (\lambda(x, a) + 1)^{\frac{3}{2}} x_n a_n} = \frac{x - \tau(a, R_h)}{(\lambda(x, a) - 1)^{\frac{1}{2}} (\lambda(x, a) + 1)^{\frac{3}{2}} x_n a_n}. \]

Proof. Applying the above lemma we compute

\[ D_{\ell} \lambda(x, a) = \sum_{i=0}^{n} e_i \left( \frac{x_i - a_i - a_n (\lambda(x, a) - 1) \delta_{in}}{x_n a_n} \right) \]
\[ = \frac{P(x - a) - (x_n - a_n \lambda(x, a)) e_n}{x_n a_n}. \]

Since

\[ \frac{|x - a|}{|x - \hat{a}|} = \sqrt{\frac{\lambda(x, a) - 1}{\lambda(x, a) + 1}} \]

we infer

\[ D_{\ell} \left( \sqrt{\frac{\lambda(x, a) - 1}{\lambda(x, a) + 1}} \right) = \frac{1}{\sqrt{\frac{\lambda(x, a) - 1}{\lambda(x, a) + 1}}^2 (\lambda(x, a) + 1)^2} D_{\ell} \lambda(x, a) \]
\[ = \frac{P(x - a) - (x_n - a_n \lambda(x, a)) e_n}{(\lambda(x, a) - 1)^{\frac{1}{2}} (\lambda(x, a) + 1)^{\frac{3}{2}} x_n a_n}. \]

The proof is complete. ■
Theorem 4.4 If $d_h(x, a)$ is the hyperbolic distance between the points $x$ and $a$ in $\mathbb{R}^{n+1}$ then

$$p(x, a) = \frac{x - \tau(a, x)}{2^n x_n a_n^{n+1} \sinh^{n+1} d_h(x, a)} = \frac{1}{2^n x_n |x - \tau(a, x)|^{n+1}},$$

where

$$\tau(a, x) = a_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n \cosh d_h(x, a) e_n.$$

Proof. Noting the definition of $p(x, a)$, see Proposition 3.3, and

$$1 - \frac{|a - x|^2}{|a - \hat{y}|^2} = \frac{4 a_n y_n}{|a - \hat{y}|^2},$$

we obtain

$$p(x, a) = -\frac{1}{2^{n-1} a_n} \mathcal{D}_\ell \left( \int_{\frac{|a-x|}{|x-a|}}^{1} \frac{(1 - s^2)^{n-1}}{s^n} ds \right)$$

$$= \frac{1}{2^{n-1} a_n} \left( 1 - \frac{|a-x|^2}{|x-a|^2} \right)^{n-1} \frac{1}{|x-a|^n} \mathcal{D}_\ell \left( \frac{|a-x|}{|x-a|} \right)$$

$$= \frac{1}{2^{n-1} a_n} \frac{(4 x_n a_n)^{n-1}}{|x-a|^{2(n-1)}} \frac{|a-x|^{-n}}{|x-a|^{-n}} \mathcal{D}_\ell \left( \frac{|a-x|}{|x-a|} \right).$$

Applying Lemma 2.2 and Lemma 2.3 we obtain

$$2^n a_n^2 p(x, a)$$

$$= \left( \frac{2 x_n a_n}{2 a_n x_n (\lambda(x, a) + 1)} \right)^{n-1} (\lambda(x, a) + 1)^{\frac{n}{2}} \frac{P(x-a) - (x_n - a_n \lambda(x, a)) e_n}{(\lambda(x, a) - 1)^{\frac{n}{2}} (\lambda(x, a) - 1)^{\frac{1}{2}} (\lambda(x, a) + 1)^{\frac{3}{2}} x_n a_n}$$

$$= \frac{P(x-a) - (x_n - a_n \lambda(x, a)) e_n}{(\lambda(x, a)^2 - 1)^{\frac{n+1}{2}} x_n a_n}.$$

Since

$$\lambda(x, a)^2 - 1 = \cosh^2 d_h(x, a) - 1 = \sinh^2 d_h(x, a)$$

we conclude

$$p(x, a) = \frac{P(x-a) - (x_n - a_n \lambda(x, a)) e_n}{2^n x_n a_n^{n+1} \sinh^{n+1} d_h(x, a)}.$$
Thus the first equality holds. Using Corollary 2.5 we obtain

\[ p(x, a) = \frac{1}{2^n x_n} \frac{x - \tau(a, x)}{|x - \tau(a, x)|^{n+1}}, \]

and the proof follows. ■

**Remark 4.5** The theorem in above give us an interpretation to the p-kernel. In the Euclidean Clifford analysis the Cauchy’s kernel is \( \frac{x - a}{|x - a|^{n+1}} \) (of course up to constant). In the hyperbolic case the p-kernel is just the Euclidean kernel, but we compute it in the different center. Also there is the coefficient \( 1/x_n \), what is something what we expected.

Our next aim is to compute the q-kernel using hyperbolic tools. We start the mission giving the following lemma.

**Lemma 4.6** Let \( \Omega \subset \mathbb{R}^{n+1} \) be an open subset. If \( g : (a_1, b_1) \times (a_2, b_2) \to \mathbb{R} \) and \( f_i : (a_i, b_i) \to \mathbb{R} \) are differentiable for \( i = 1, 2 \). Assume that \( h_i(\Omega) \subset (a_i, b_i) \) for \( i = 1, 2 \). Then

\[ D_{\ell}(g \circ (h_1, h_2)) = \partial_1 g \circ (h_1, h_2) D_{\ell} h_1 + \partial_2 g \circ (h_1, h_2) D_{\ell} h_2. \]

As an application of the preceding lemma we obtain the following theorem.

**Theorem 4.7** If \( d_h(x, a) \) is the hyperbolic distance between the points \( x \) and \( a \) in \( \mathbb{R}^{n+1}_+ \) then

\[ q(x, a) = \frac{(x - \tau(a, x)) \cosh d_h(x, a) - a_n \sinh^2 d_h(x, a)e_n}{(2a_n x_n)^n \sinh^{n+1} d_h(x, a)} \]

\[ = \frac{1}{(2x_n)^n} \frac{x - \tau(a, x)}{|x - \tau(a, x)|^{n+1}} Q \tau(a, x) - \frac{1}{(2x_n)^n} \frac{1}{|x - \tau(a, x)|^{n-1}} e_n, \]

where

\[ \tau(a, x) = a_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n \cosh d_h(x, a)e_n. \]

**Proof.** Recall

\[ q(x, a) = \frac{1}{2a_n x_n} D_{\ell} H(x, a) \]

where

\[ H(x, a) = \frac{1}{((2x_n a_n)(\lambda(x, a) - 1))^{n+1} ((2x_n a_n)(\lambda(x, a) + 1))^{n+1}}. \]
Let us define the functions $g(s_1, s_2) = \left(\frac{1}{s_1 s_2}\right)^{\frac{n-1}{2}}$, and $h_i(x) = (2x_1 a_n)(\lambda(x,a) + (-1)^i)$ for $i = 1, 2$. Thus $H = g \circ (h_1, h_2)$. Now

$$\partial_i g(s_1, s_2) = -\frac{n-1}{2} \frac{1}{s_1(s_1 s_2)^{\frac{n-1}{2}}}.$$ 

and using Lemma 4.3

$$\overline{D}_i h_i(x) = 2a_n x \overline{D}_i \lambda(x,a) + 2a_n(\lambda(x,a) + (-1)^i) \overline{D}_i x_n$$

$$\overline{D}_i(g(h_1(x), h_2(x))) = \partial_1 g \circ (h_1(x), h_2(x)) \overline{D}_i h_1(x) + \partial_2 g \circ (h_1(x), h_2(x)) \overline{D}_i h_2(x)$$

$$= -\frac{n-1}{2} \frac{\overline{D}_i h_1(x)}{h_1(x) h_2(x)^{\frac{n-1}{2}}} - \frac{n-1}{2} \frac{\overline{D}_i h_2(x)}{h_2(x) h_1(x)^{\frac{n-1}{2}}}$$

$$= -\frac{n-1}{2} \frac{h_1(x) \overline{D}_i h_1(x) + h_1(x) \overline{D}_i h_2(x)}{(h_1(x) h_2(x))^{\frac{n-1}{2}}}.$$ 

Consider now the numerator of the above quotient. We compute

$$\sum_{i=1}^{2} h_i(x) \overline{D}_i h_{3-i}(x) = 4x_1 a_n \sum_{i=1}^{2} (\lambda(x,a) + (-1)^i)((x - \tau(a,x)) - a_n(\lambda(x,a) + (-1)^{3-i})e_n)$$

$$= 8x_1 a_n \lambda(x,a)(x - \tau(a,x))$$

$$- 4x_1 a_n^2 \sum_{i=1}^{2} (\lambda(x,a) + (-1)^i)(\lambda(x,a) - (-1)^i)e_n.$$

The second term is

$$4x_1 a_n^2 \sum_{i=1}^{2} (\lambda(x,a) + (-1)^i)(\lambda(x,a) - (-1)^i)e_n$$

$$= 4x_1 a_n^2 \sum_{i=1}^{2} (\lambda^2(x,a) - 1)e_n$$

$$= 8x_1 a_n^2 (\lambda^2(x,a) - 1)e_n.$$ 

Thus

$$h_2(x) \overline{D}_i h_1(x) + h_1(x) \overline{D}_i h_2(x) = 8x_1 a_n \lambda(x,a)(x - \tau(a,x)) - 8x_1 a_n^2 (\lambda^2(x,a) + 1)e_n.$$
Hence we obtain
\[ q(x,a) = \frac{1}{4} \left( h_2(x) \mathcal{D}_t h_1(x) + h_1(x) \mathcal{D}_t h_2(x) \right) \frac{\lambda(x,a)}{(\lambda(x,a))^2} \]
\[ = \frac{1}{4} \left( 8x_n a_n (x - \tau(a,x)) - 8x_n a_n^2 (\lambda^2(x,a) - 1) e_n \right). \]

Since \((\lambda(x,a) - 1)(\lambda(x,a) + 1) = \lambda(x,a)^2 - 1 = \sinh^2 d_h(x,a)\) we have
\[ q(x,a) = \frac{(x - \tau(a,x)) \cosh d_h(x,a) - a_n \sinh^2 d_h(x,a) e_n}{(2x_n a_n)^{n+1} \sinh^{n+1} d_h(x,a)}, \]
and the first equality holds. Then
\[ q(x,a) = \frac{(x - \tau(a,x)) \cosh d_h(x,a)}{(2x_n a_n)^n \sinh^n d_h(x,a)} - \frac{e_n}{(2x_n a_n)^{n-1} \sinh^{n-1} d_h(x,a)}. \]
Since \(\cosh d_h(x,a) = \frac{Q_\tau(a,x)}{a_n}\) we have
\[ q(x,a) = \frac{(x - \tau(a,x))}{(2x_n a_n)^n \sinh^n d_h(x,a)} Q_\tau(a,x) - \frac{e_n}{(2x_n a_n)^{n-1} \sinh^{n-1} d_h(x,a)}. \]
The second equality follows from Corollary 2.5. ■

**Remark 4.8** The theorem in above give us an interpretation to the \(q\)-kernel. Recall that the Newton’s kernel in the theory of harmonic function is (up to constant) \(\frac{1}{|x-a|^{n-1}}\). Thus we see that the \(q\)-kernel is a linear combination of the Cauchy’s and the Newton’s kernels with the center \(\tau(a,x)\). Moreover, since we consider the kernel of the Cauchy’s formula for the \(Q\)-part of a hypermonogenic function, it can be expected that the coefficient \(e_n\) is in the special role.

5 The Mean-Value Theorem for the \(P\)-Part of a Hypermonogenic Function

Lastly we will study a mean-value property of the \(P\)-part of a hypermonogenic function. The hyperbolic machinery developed in the preceding sections has a strong influence on the proof.
Theorem 5.1 Let \( \Omega \) be an open subset of \( \mathbb{R}^{n+1}_+ \). If \( f \) is hypermonogenic in \( \Omega \), then
\[
P f(a) = \frac{a_n^n}{\omega_{n+1} R_e^n} \int_{\partial B_h(a,R_h)} P f(x) \frac{dS(x)}{a_n^n}
\]
for any hyperbolic ball \( B_h(a,R_h) \) with \( \overline{B_h(a,R_h)} \subset \Omega \).

Proof. Using Proposition 3.3 we obtain
\[
P f(a) = 2^n a_n^n \int_{\partial B_h(a,R_h)} P(p(x,a) \nu(x)f(x)) \frac{dS(x)}{x_n^{n-1}}.
\]

Since
\[
p(x,a) = \frac{x - \tau(a)}{2^n x_n a_n^{n+1} \sinh^{n+1} d_h(x,a)},
\]
and
\[
\nu(x) = \frac{x - \tau(a)}{R_e}
\]
we have
\[
p(x,a) \nu(x) = \frac{|x - \tau(a)|^2}{2^n x_n a_n^{n+1} R_e \sinh^{n+1} d_h(x,a)}.
\]
Moreover since \( |x - \tau(a)|^2 = R_e^2 \) and \( R_e = a_n \sinh R_h \) we obtain
\[
p(x,a) \nu(x) = \frac{1}{2^n x_n R_e^n}.
\]

Then
\[
P f(a) = \frac{a_n^n}{\omega_{n+1} R_e^n} \int_{\partial B_h(a,R_h)} P f(x) \frac{dS(x)}{a_n^n}.
\]
The proof is complete. \( \blacksquare \) Also we’d like to recall the following structure theorem.

Theorem 5.2 ([6]) Let \( U \subset \mathbb{R}^{n+1}_+ \) be open. The following properties are equivalent:

(a) \( h \) is hyperbolically harmonic on \( U \).

(b) \( h \) is smooth and
\[
h(a) = \frac{1}{\sigma_n \sinh^n R_h} \int_{\partial B_h(a,R_h)} h(x) d\sigma_h(x)
\]
for all \( \overline{B_h(a,R_h)} \subset U \).
(c) \( h \) is smooth and
\[
h(a) = \frac{1}{V(B_h(a, R_h))} \int_{B_h(a, R_h)} h(x) dx_h(x)
\]
for all \( \overline{B_h(a, R_h)} \subset U \) where \( V(B_h(a, R_h)) = \sigma_n \int_0^{R_h} \sinh^n(t) dt \) is the hyperbolic volume of the ball \( B_h(a, R_h) \).

The corresponding theorem is available also for the \( Q \)-part of a hypermonogenic function. We shall study these topics in the forthcoming paper [13].

References


