MAT-61506: Dynamical Systems and Chaos
Part II: Biology Applications

Lecture 1: Introduction + 1D models

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Preliminary notes

- Semi-lecture mode: some elements of a seminar
- Wed 10–12 and Fri 10–12/12–14. The lecture room is TD308.
- The lecturer of Part II: Biology Applications of this course is available almost any time at TD325. You might want to write an e-mail as well.
- One has to fulfill the project work in order to pass the course. The project involves a little of research work, thus, requiring some creativity.
- The possible projects are listed in the web-site of the course (http://matriisi.ee.tut.fi/courses/MAT-35006/index.html). Not all the projects are biology driven.
Biology Applications

- The main theme is Biology.
- Example dynamical systems are to be analyzed in this perspective.
- One must be familiar with the Differential Equations and some basic techniques for analyzing them.
Lecture 1  1D dynamical systems represented in terms of ODE’s. Simple means of qualitative analysis. Lyapunov stability analysis. Notion of bifurcations.


Lecture 3  2D linear systems. Law of mass action.

Lecture 4  2D nonlinear systems. Lyapunov’s stability notion for 2D systems. Lotka-Volterra type equations.
Detailed Content

Lecture 5  Molecular systems. Michaelis-Menten paradigm and notion of the scale of processes. Bistable systems.

Lecture 6  Oscillations in life.

Lecture 7  Excitability phenomena in cells. Neurons and neuronal dynamics.

Lecture 8  Coupled systems. “Population” of dynamical systems. Some more advanced examples.
The notion comprises movement over time, or evolution (however, the term implies some progress as well).

In order to see the movement one must define the space where the system’s movement is realized. Thus, the notion also comprises the phase space – all possible states that the system can (potentially) occupy.

The movement over time in the phase space is governed by the law of evolution. In general, the dynamical systems theory assumes the deterministic operator of evolution.
Living things are dynamical systems.

- Dynamical systems theory can be applied to a very broad range of phenomena.
- Biological systems are not exceptions.
- However, they possess certain properties that must be accounted for:
  1. **Complex systems**: many components, spatial organization etc. Aggregated approach (population dynamics) or detailed modelling.
  2. **Systems with reproduction**: auto-catalytic characteristics. Tendency to avoid global equilibrium.
  3. **Open system**: always interact with the environment by interchanging matter and energy.
  4. **Hierarchy of regulation**: regulated by the complex multi-level regulatory mechanisms. Heterogeneous systems.
Example: growth of a population

The Malthus model of exponential (non-limited) growth.

\[ \frac{dx}{dt} = rx, \]

where \( r = R - S \), \( R \) – reproduction rate, \( S \) – senescence rate. The solution of the exponential growth model is:

\[ x(t) = C_0 e^{rt}, \]

\( C_0 \) is a constant.
Malthus model dynamics

Figure: The dynamics of the Malthus model depending on the coefficient \( r \) for a single initial condition.
The growth of a population with the limitation

\[ \frac{dx}{dt} = r x \left( 1 - \frac{x}{K} \right), \]

where \( K \) and \( r \) are positive constants.

The solution of the equation is:

\[ x(t) = \frac{x_0 K e^{r \cdot t}}{K - x_0 + x_0 e^{r \cdot t}}, \]

where \( x_0 \) is the initial condition, i.e. \( x_0 = x(t = 0) \).
Verhulst model dynamics.

Figure: The Verhulst equation dynamics depending on the initial conditions $x_0$. $K$ is the population capacity. $K = 1$, $r = 1$. 
Steady state analysis

- For the qualitative exploration of 1D systems, it is enough to analyze the RHS function and the phase space.
- Consider equation
  \[ x' = f(x) \]
- \( f(x) = 0 \) gives the vector of solutions \( \bar{x}^* \).
- Solutions \( \bar{x}^* \) are the points at which \( x \) does not change over time since \( \frac{dx}{dt} = 0 \), i.e. \( \bar{x}^* \) are the resting points of the system or steady states.
- However, there are different kinds of the steady states that exist.
Analyzing the dynamics: one equation

\[ \frac{dx}{dt} = f(x) \]

\(X_i^*\) (dashed lines) is the steady states; solid lines are integral curves.
Stability analysis

- **Stable** steady states, or **sinks** that the trajectory (i.e. function $x(t)$) tends to in forward time.
- **Unstable** steady states, or **sources** that the trajectory $x(t)$ tends to in backward time.
- **Saddle** point, that is neither sink nor source. However, saddle point is unstable.

Which system is stable?

*Figure*: The notion of stability according to Lyapunov.
Some more examples.

Non-biology examples for training

1. $x' = x^3 - 3x$
2. $x' = x^4 - x^2$
3. $x' = |1 - x^2|$
Changes in parameters of a dynamical system may lead to a significant **qualitative** change of the phase space of the system.

Such a change is called **bifurcation**.

The parameter value at which the bifurcation takes place is called the bifurcation point.

The number of parameters needed to be changed for a bifurcation to occur is called the **codimension** (codim) of the bifurcation.
Simple codim-1 bifurcation

Consider the system

\[ \dot{x} = f(x) \equiv \alpha - x^2 \]

For different \( \alpha \) system has different number of steady state solutions and the phase portrait of the system changes.
Codim-2 bifurcation: cusp bifurcation

\[ \dot{x} = \alpha_1 + \alpha_2 x + x^3 \]

For negative and positive \(\alpha_2\) the system has considerably different phase portraits.
We have considered simple and effective pen-and-paper techniques to analyze the dynamical behavior of 1D systems. Phase space (line in case of 1D) analysis is proven to be highly productive method for qualitative assessments on dynamics. Upon the varying parameter values dynamical systems undergo bifurcations—considerable topological changes of the phase space. We have considered the set of rough aggregated models of biological populations that are deemed to be historically earliest models of biological systems. The population is described only by the number of species it comprises.
Useful resources

- The main source for the course:

- Bifurcation theory is described here: