Chaos in Hamiltonian systems

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Course material:
Chapter 7 from Ott 1993/2002, Chaos in Dynamical Systems, Cambridge
http://matriisi.ee.tut.fi/courses/MAT-35006

Useful reading:
Goldstein 2002, Classical Mechanics, Addison Wesley

Advanced:
Lichtenberg & Lieberman 1992, Regular and Chaotic Dynamics, Springer
Arnold 1989, Mathematical Methods of Classical Mechanics, Springer

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1 Introduction: classical mechanics

Newtonian mechanics

Consider a particle of mass $m$, subject to a force field $F$, in a $d$-dimensional Euclidean space (one-body system). Newton’s second law; $m\ddot{r} = F \implies$ a system of $2d$ first order ODEs for position $r \in \mathbb{R}^d$ and velocity $\dot{r} \in \mathbb{R}^d$. These are equations of motion in phase space (the space $\mathbb{R}^d \times \mathbb{R}^d$ where $(r, \dot{r})$ are coordinates). The number of degrees of freedom of the mechanical system is $N = d$.

Example (harmonic oscillator). Set $d = 1$, $F = -kr \implies \ddot{r} = -kr/m$ [draw a picture].

\[
\begin{align*}
\frac{dr}{dt} &= \dot{r} \\
\frac{d\dot{r}}{dt} &= -\frac{k}{m} r.
\end{align*}
\]

The solution for initial values $r(0) = r_0$, $\dot{r}(0) = 0$ is $r(t) = r_0 \cos \left( t\sqrt{k/m} \right)$. [Characteristic equation $z^2 + k/m = 0 \implies z_{1,2} = \pm i\sqrt{k/m} = \pm i\omega_h \implies r(t) = a \cos \omega_h t + b \sin \omega_h t$.]

Lagrangian mechanics

For an unconstrained system of $n$ bodies: $N = dn$ (2N ODEs). Under holonomic constraints $f_j(r_1, r_2, \ldots, r_n, t) = 0$, $j = 1, \ldots, k$ we can define generalized coordinates $q_i$, $i = 1, \ldots, N$, where $N = dn - k$, using transformation equations

\[
\begin{align*}
    r_1 &= r_1(q_1, q_2, \ldots, q_N, t) \\
    \vdots \\
    r_n &= r_n(q_1, q_2, \ldots, q_N, t). 
\end{align*}
\]

Definition. The Lagrangian (function) is $L = T - V$, where $T = \sum_{i=1}^n m_i (\dot{r}_i \cdot \dot{r}_i)/2$ is the kinetic energy, and $V = V(r_1, r_2, \ldots, r_n, t)$ is the potential energy.
Through the transformation equations (1), we have $L = L(q, \dot{q}, t)$. The Hamilton’s principle

$$\delta I = 0, \quad I = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt$$

is a variational equation for finding a path $q(t)$ from $t_1$ to $t_2$ for which the line integral $I$ (action) is stationary [draw a picture]. Solution [e.g., Goldstein] yields the (Euler-)Lagrange equations of motion:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \ldots, N.$$

[For all possible paths, the system takes the one requiring the least action.]

**Example** (harmonic oscillator). $L = T - V = m\dot{r}^2/2 - kr^2/2$,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m\ddot{r}, \quad \frac{\partial L}{\partial r} = -kr \implies \ddot{r} = -\frac{k}{m}r.$$

**Hamiltonian mechanics**

**Definition.** The conjugate momenta $p_i = \partial L / \partial \dot{q}_i$ and Hamiltonian $H = \dot{q}_i p_i - L$ (Einstein summation convention).

We do a Legendre transformation;

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt = \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt$$

Since we wish $H = H(q, p, t)$,

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt.$$

Equating terms, we have the Hamiltons equations of motion:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{2}$$

The $(q, p) \in \mathbb{R}^N \times \mathbb{R}^N$ are canonical phase-space variables.
Example (harmonic oscillator). \( p = \partial L / \partial \dot{r} = m \dot{r} \), \( H = m \dot{r}^2 - L = m \dot{r}^2 / 2 + kr^2 / 2 = T + V = E \).

In general, if the transformation equations (1) do not depend on \( t \) explicitly, and the forces are conservative (of the form \( F = -\nabla \Phi \)), Hamiltonian is the total energy.

For a time-independent Hamiltonian \( H(p, q) \) the total energy is conserved:

\[
\frac{dH}{dt} = \frac{\partial H}{\partial q} \dot{q} + \frac{\partial H}{\partial p} \dot{p} = \frac{\partial H}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial H}{\partial q} = 0.
\]

\( \Rightarrow H(p, q) = E \) is a constant of motion.

2 Symplectic structure

The class of Hamiltonian systems is very special. Let \( x = (p, q)^T \) and

\[
f(x, t) = \Omega \left( \frac{\partial H}{\partial x} \right)^T, \quad \text{where} \quad \Omega = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.
\]

Now, Hamilton’s equations are \( \dot{x} = f(x, t) \). Note that \( f(x, t) \in \mathbb{R}^{2N} \) (the Hamiltonian vector field) is fully determined by \( H(x, t) \in \mathbb{R} \).

Theorem (Liouville’s theorem). Hamilton’s equations preserve \( 2N \)-dimensional volumes.

Proof.

\[
\frac{\partial}{\partial x} \cdot f = \frac{\partial}{\partial p} \cdot \left( -\frac{\partial H}{\partial q} \right) + \frac{\partial}{\partial q} \cdot \left( \frac{\partial H}{\partial p} \right) = 0,
\]

\[
\frac{d}{dt} \int_V d^{2N}x = \oint_S \frac{dx}{dt} \cdot dS = \oint_S f \cdot dS = \int_V \left( \frac{\partial}{\partial x} \cdot f \right) d^{2N}x = 0.
\]

[ differentiation under integral sign, Hamilton’s equations, divergence (Gauss’) theorem.]

Corollary. There are no attractors in Hamiltonian systems.
Hamilton’s equations are \textit{symplectic}. Consider three orbits \((p, q)\), \((p + \delta p, q + \delta q)\), and \((p + \delta p', q + \delta q')\). The differential symplectic area is the sum of parallelogram areas: [Ott, fig. 7.1]

\[
\sum_{i=1}^{N} \left| \begin{array}{cc} \delta p_i & \delta q_i \\ \delta p'_i & \delta q'_i \end{array} \right| = \sum_{i=1}^{N} (\delta p_i \delta q'_i - \delta q_i \delta p'_i) = \delta p \cdot \delta q' - \delta q \cdot \delta p' = \delta x^T \Omega \delta x'.
\]

\[
\frac{d}{dt} (\delta x^T \Omega \delta x') = \frac{d \delta x^T}{dt} \Omega \delta x' + \delta x^T \frac{d \delta x'}{dt} = \left( \frac{\partial f}{\partial x} \right)^T \Omega \delta x' + \delta x^T \Omega \frac{\partial f}{\partial x} \delta x'
\]

\[
= \delta x^T \left[ \left( \frac{\partial f}{\partial x} \right)^T \Omega + \Omega \frac{\partial f}{\partial x} \right] \delta x'
\]

\[
= \delta x^T \left[ \left( \frac{\partial^2 H}{\partial x^2} \right)^T \Omega + \Omega \frac{\partial^2 H}{\partial x^2} \right] \delta x'
\]

\[
= 0.
\]

[linearization of \(f(x + \delta x, t)\), \(\Omega \Omega = -I\), \(\Omega^T = -\Omega\), \(\partial^2 H/\partial x^2\) symmetric]. Symplectic condition \(\Rightarrow\) conservation of volume.

\textbf{Definition.} The differential symplectic area is a differential form of \textit{Poincaré’s integral invariant}

\[
\oint_{\gamma} p \cdot dq = \sum_{i=1}^{N} \oint_{\gamma} p_i dq_i,
\]

where \(\gamma\) is a closed path in \((p, q)\) at constant \(t\).

A generalization in extended \(2N + 1\) phase space \((p, q, t)\) [Ott, fig. 7.2]:

\textbf{Theorem} (Poincaré-Cartan integral theorem).

\[
\oint_{\Gamma_1} (p \cdot dq - H dt) = \oint_{\Gamma_2} (p \cdot dq - H dt),
\]

where \(\Gamma_1\) and \(\Gamma_2\) are paths around the ‘tube’ of trajectories.
Γ₁ and Γ₂ at constant times (dt = 0) → Poincaré’s integral invariant. If \( H = H(p, q) \) and Γ₁ and Γ₂ are on this surface, we have \( \oint H \, dt = 0 \) and

\[
\oint_{Γ₁} p \cdot dq = \oint_{Γ₂} p \cdot dq,
\]

where Γ₁ and Γ₂ does not need to be at constant t.

### 3 Canonical changes of variables

Let us define a new set of phase-space variables \( X = (P, Q)^T \) by using a transformation \( g : \mathbb{R}^{2N} \to \mathbb{R}^{2N}, x \mapsto X \).

**Definition.** The transformation \( g \) is canonical, if it preserves the differential symplectic area;

\[
\delta p \cdot \delta q' - \delta q \cdot \delta p' = \delta P \cdot \delta Q' - \delta Q \cdot \delta P'.
\] (3)

Canonical transformations are typically performed by using a generating function, e.g., \( S = S(P, q, t) \);

\[
Q = \frac{\partial S}{\partial P},
\]
\[
p = \frac{\partial S}{\partial q},
\]

which gives \( (P, Q) \) implicitly. The transformed Hamiltonian is \( K(P, Q, t) = H(p, q, t) + \partial S/\partial t \). [Can be derived using the Hamilton’s principle; see Goldstein.] In order to check the canonicity of the above one can substitute

\[
\delta Q = \frac{\partial^2 S}{\partial P^2} \delta P + \frac{\partial^2 S}{\partial P \partial q} \delta q,
\]
\[
\delta p = \frac{\partial^2 S}{\partial P \partial q} \delta P + \frac{\partial^2 S}{\partial q^2} \delta q,
\]

into the condition (3) [homework].

The new variables satisfy Hamilton’s equations: \( dX/dt = \Omega(\partial K/\partial X)^T \). Hence, the underlying symplectic structure of a given Hamiltonian system is invariant, and can be represented using any suitable choice of canonical phase-space variables.
4 Hamiltonian maps

Consider the map $\mathcal{M}_h : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$:

$$\mathcal{M}_h(x(t), t) = x(t + h).$$

Its differential variation [linearize $\mathcal{M}_h(x + \delta x, t)$] is

$$\frac{\partial \mathcal{M}_h}{\partial x} \delta x(t) = \delta x(t + h).$$

Symplectic condition $\implies$

$$\delta x^T(t)\Omega \delta x'(t) = \delta x^T(t + h)\Omega \delta x'(t + h)$$

$$= \left(\frac{\partial \mathcal{M}_h}{\partial x} \delta x(t)\right)^T \Omega \left(\frac{\partial \mathcal{M}_h}{\partial x} \delta x'(t)\right)$$

$$= \delta x^T(t) \left(\frac{\partial \mathcal{M}_h}{\partial x}\right)^T \Omega \left(\frac{\partial \mathcal{M}_h}{\partial x}\right) \delta x'(t)$$

$$\implies \Omega = \left(\frac{\partial \mathcal{M}_h}{\partial x}\right)^T \Omega \left(\frac{\partial \mathcal{M}_h}{\partial x}\right)$$

We say, $\partial \mathcal{M}_h / \partial x$ is symplectic. Generally, matrix $A$ is symplectic, if $\Omega = A^T \Omega A$. The product of symplectic matrices $A$ and $B$ is symplectic;

$$(AB)^T \Omega (AB) = B^T (A^T \Omega A) B = B^T \Omega B = \Omega.$$ 

Theorem (Poincaré recurrence theorem). Examine a Hamiltonian $H(p, q)$ with orbits bounded in a finite subset $D \subset \mathbb{R}^{2N}$. Let a ball $R_0 \in D$ have radius $\varepsilon > 0$. In time, some of the orbits leaving $R_0$ will return to it.

Proof. Evolve $R_0$ with the volume-preserving $\mathcal{M}_h$, obtaining subsequent regions $R_1, R_2, \ldots$. Conclude that $\exists R_r \cap R_s \neq \emptyset \implies R_{r-s} \cap R_0 \neq \emptyset$, $r > s$ [Arnold, fig. 51].

Poincaré maps

In general, a Poincaré map gives an intersection of an orbit with a lower-dimensional subspace of the phase space, called Poincaré section (or surface of section, SOS). For periodic orbits, plotting subsequent intersections typically draws a figure which reflects deeper characteristics of the dynamical system.
Definition. *Poincaré section* for a Hamiltonian system $H(p,q,t)$ where $H$ is $\tau$-periodic in time:

- extended $(2N + 1)$-dimensional phase space $(p,q,\xi)$, where $\xi = t$,
- additional equation $d\xi/dt = 1$,
- let $\xi' = \xi \mod \tau$,
- set $\xi' = t_0 \in [0,\tau)$ as the $2N$-dimensional SOS.

Because of the periodicity, $\mathcal{M}_\tau(x, t_0) = \mathcal{M}_\tau(x, t_0 + k\tau)$, $k \in \mathbb{Z}$. The surface of section map $x_{n+1} = M(x_n) = \mathcal{M}_\tau(x_n, t_0)$ is symplectic.

Example (*Standard map*). The ‘kicked rotor’: bar of moment of inertia $I$ and length $l$, frictionless pivot [Ott, fig. 7.3]. Vertical impulse of strength $K/l$ at times $t = 0, \tau, 2\tau, \ldots$ Canonical variables $(p_\theta, \theta)$, Hamiltonian

$$H(p_\theta, \theta, t) = \frac{p_\theta^2}{2I} + K \cos \theta \sum_n \delta(t - n\tau),$$

where $\delta$ denotes the Dirac delta, and equations of motion (2)

$$\frac{dp_\theta}{dt} = K \sin \theta \sum_n \delta(t - n\tau),$$
$$\frac{d\theta}{dt} = \frac{p_\theta}{I}.$$

SOS: $(p_\theta, \theta)$ after each kick. Integration over $t \in (n\tau, (n + 1)\tau]$ yields $p_{n+1} - p_n = K \sin \theta_{n+1}$ and $\theta_{n+1} - \theta_n = p_n \tau / I$. Setting $\tau / I = 1$, we have

$$p_{n+1} = p_n + K \sin \theta_{n+1},$$
$$\theta_{n+1} = (\theta_n + p_n) \mod 2\pi.$$

The mapping $(N = 1)$ preserves differential areas, if

$$\det(\delta x_n, \delta x'_n) = \det \left[ (\partial x_{n+1}/\partial x_n)(\delta x_n, \delta x'_n) \right],$$

where $x_n = (p_n, \theta_n)^T$. Symplecticity can hence be verified by calculating

$$\det \begin{bmatrix}
\partial p_{n+1}/\partial p_n & \partial p_{n+1}/\partial \theta_n \\
\partial \theta_{n+1}/\partial p_n & \partial \theta_{n+1}/\partial \theta_n
\end{bmatrix} = \det \begin{bmatrix}
1 + K \cos \theta_{n+1} & K \cos \theta_{n+1} \\
1 & 1
\end{bmatrix} = 1.$$
**Definition.** Poincaré section for a Hamiltonian system $H(p,q)$:

- motion in a $(2N-1)$-dimensional energy surface $H(p,q) = E$,
- by choosing, e.g., $q_1 = 0$, we determine a $(2N-2)$-dimensional SOS.

For a unique SOS mapping $x_{n+1} = M(x_n)$, we often need an additional constraint, e.g., $p_1 > 0$.

**Example (Logarithmic potential).** The potential $\Phi$ near the centre of a flattened galaxy can be modelled as

$$\Phi(x, y) = \frac{1}{2} \ln \left( x^2 + \frac{y^2}{a^2} + b^2 \right),$$

where $a, b \in \mathbb{R}$ are parameters [draw a picture]. For a star moving in this potential, the Hamiltonian per unit mass is

$$H(p_x, p_y, x, y) = \frac{1}{2} \left[ p_x^2 + p_y^2 + \ln \left( x^2 + \frac{y^2}{a^2} + b^2 \right) \right].$$

Equations of motion (2) become

$$\dot{p}_x = -\frac{x}{x^2 + y^2/a^2 + b^2},$$

$$\dot{p}_y = -\frac{y}{a^2(x^2 + y^2/a^2 + b^2)},$$

$$\dot{x} = p_x,$$

$$\dot{y} = p_y.$$

Choose $y = 0$ as the SOS. Now

$$H(p_x, p_y, x, 0) = E \implies p_y^2 = 2E - p_x^2 - \ln \left( x^2 + b^2 \right).$$

Hence, there are two points $\pm p_y$ corresponding to $y = 0$. We choose that $p_y > 0$ on the SOS. In this case, we cannot solve the equations of motion analytically, and points $(p_x, x)$ on the SOS have to be determined by numerical integration.
5 Integrable systems

Consider a Hamiltonian $H(p, q)$. A constant of motion is a quantity that does not change when the system evolves in time, e.g., $H(p, q) = E$. In general, for a function $f(p(t), q(t))$, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial p} \dot{p} + \frac{\partial f}{\partial q} \dot{q} = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}.$$ 

If $f(p, q)$ is a constant of motion, we can write $[f, H] = 0$, where the Poisson bracket for functions $f$ and $g$ is defined as

$$[f, g] := \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}.$$ 

Poisson bracket is anticommutative: $[f, g] = -[g, f]$.

**Theorem** (Liouville’s theorem on integrable systems). The Hamiltonian system $H(p, q)$ is integrable, if it has $N$ independent constants of motion $f_i(p, q)$, $i = 1, \ldots, N$ which are in involution, i.e., $[f_i, f_j] = 0$, $\forall i, j$.

[Proof: see, e.g., Arnold.] The independent constants of motion restrict an integrable system on an $N$-dimensional surface in the phase space. Because the constants are in involution, any bounded orbit on this surface is topologically equivalent to an $N$-dimensional torus [Ott, fig. 7.4a].

**Action-angle variables**

The action-angle variables provide an explicit transformation between points $(p, q)$ and points on the $N$-torus. Through a canonical change of variables, it is possible to transform to coordinates $(P, Q)$ in which the new momenta $P_i(p, q)$, $i = 1, \ldots, N$ are constants of motion. In such a case $dP_i/dt = -\partial H/\partial Q_i = 0 \Rightarrow H = H(P)$.

A particularly convenient choice is

$$P_i = J_i := \frac{1}{2\pi} \oint_{\gamma_i} p \cdot dq,$$

where the paths $\gamma_i$ wrap around each possible angle direction $i = 1, \ldots, N$ on the $N$-torus [Ott, fig. 7.4b]. The new variables $J_i$ are actions. They are constants of motion by the Poincaré-Cartan integral theorem.
The corresponding conjugate coordinates $Q_i = \theta_i$ are angles. Suppose that action-angle variables $(J, \theta)$ are obtained by the generating function $S(J,q)$:

$$\theta = \frac{\partial S}{\partial J},$$

$$p = \frac{\partial S}{\partial q}$$

By integrating the latter equation around $\gamma_i$, we examine the change in $S$:

$$\Delta_i S = \oint_{\gamma_i} p \cdot dq = 2\pi J_i.$$

Then,

$$\Delta_i \theta = \frac{\partial}{\partial J} \Delta_i S = 2\pi \frac{\partial}{\partial J} J_i,$$

or

$$\Delta_i \theta_j = 2\pi \delta_{ij},$$

where $\delta_{ij} = 1$, if $i = j$, and zero otherwise. Therefore, after one circuit around $\gamma_i$, the angle $\theta_i$ increases by $2\pi$, and the other angles return to their original values.

The new Hamiltonian is $H = H(J)$, and

$$\frac{dJ}{dt} = 0,$$

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial J} =: \omega(J).$$

A solution from initial values $(J_0, \theta_0)$ is $J(t) = J_0$ and $\theta(t) = \theta_0 + \omega t$. The constants of motion $\omega$ are frequencies which can be interpreted as angular velocities on the torus.

**Definition.** An orbit on the torus is quasiperiodic if there is no vector of integers $m \in \mathbb{Z}^n$ such that

$$m \cdot \omega = 0.$$

A quasiperiodic orbit fills the surface of the torus as $t \to \infty$. On the other hand, if $\omega_i/\omega_j \in \mathbb{Q}, \forall i,j$, the orbit is periodic, and closes on itself. The set of tori with periodic orbits is dense in the phase space, but has a zero Lebesgue measure. Hence, orbits on a randomly picked torus are quasiperiodic with probability of one.
Example (harmonic oscillator). In general, for \( N = 1 \), we have

\[
H(p, q) = \frac{p^2}{2m} + V(q),
\]

where \( p, q \in \mathbb{R} \), and \( V(q) \) is a potential energy of a general form [Ott, fig. 7.5]. We have

\[
J = \frac{1}{\pi} \int_{q_1}^{q_2} \{2m[E - V(q)]\}^{1/2} dq.
\]

For the harmonic oscillator, \( V(q) = m\omega_h^2 q^2/2 \), where \( \omega_h = \sqrt{k/m} \) is the angular velocity of the oscillation, and \( k \) is the spring constant. From \( V(q) = E \) we have \( q_2 = -q_1 = [2E/(m\omega_h^2)]^{1/2} \), and integration 1 yields \( J = E/\omega_h \). Thus,

\[
H(J) = \omega_h J,
\]

and \( \omega(J) = \partial H/\partial J = \omega_h \), and is (in this rare case) independent of \( J \). The angle \( \theta(t) = \theta_0 + \omega_h t \). In order to use the generating function \( S(J, q) \), we write \( H(p, q) = H(J) \), and substitute \( p = \partial S/\partial q \), obtaining

\[
\frac{\partial S}{\partial q} = \sqrt{2m(\omega_h J - m\omega_h^2 q^2/2)}
\]

Solving for \( \theta \), and integrating 2, gives

\[
\theta = \frac{\partial S}{\partial J} = \frac{m\omega_h}{\sqrt{2m(\omega_h J - m\omega_h^2 q^2/2)}} \int [2m(\omega_h J - m\omega_h^2 q^2/2)]^{-1/2} dq
\]

\[
= \int \left( \frac{2J}{m\omega_h} - q^2 \right)^{-1/2} dq
\]

\[
= \arcsin \left( q \sqrt{\frac{m\omega_h}{2J}} \right),
\]

and we have \( (p, q) \) as a function of \( (J, \theta) \);

\[
q = \sqrt{\frac{2J}{m\omega_h}} \sin \theta,
\]

\[
p = \sqrt{2Jm\omega_h \cos \theta}.
\]

---

1 \( \int (a^2 - x^2)^{1/2} dx = \frac{a}{2} (a^2 - x^2)^{1/2} + \frac{a^2}{2} \arcsin \frac{x}{a} \)

2 \( \int \frac{dx}{(a^2 - x^2)^{1/2}} = \arcsin \frac{x}{a} \)
The trajectory \((p(\theta), q(\theta))\) is an ellipse [Ott, fig. 7.6]. In general, the mapping \((p, q) \mapsto (J, \theta)\) for \(N = 1\) transforms a closed curve into a one-dimensional torus (a circle).

For \(N > 1\), a general procedure for finding the action-angle variables \((J, \theta)\) involves solving the Hamilton-Jacobi equation

\[
H\left(\frac{\partial S}{\partial q}, q\right) = E.
\]

If we are lucky, the equation is solvable by separation of variables, and we have \(N\) separation constants which are also constants of motion. The motion is typically quasiperiodic.

### 6 Perturbations and the KAM theorem

The phase space of integrable Hamiltonian systems is foliated by tori. These appear as nested closed curves on the SOS. The existence of \(N\) integrals of motion allows us to access the tori via action-angle variables.

We know how to solve the Hamilton-Jacobi equation \((N > 1)\) for only a handful of systems, e.g., harmonic oscillator, isochrone, and general Stäckel potentials. On the other hand, the logarithmic potential, for example, numerically demonstrates the existence of an additional constant of motion (besides \(E\)) almost everywhere on the SOS [computer demo, logarithmic potential].

Motion on a torus is regular, and chaotic orbits are possible only in regions of phase space where invariant tori do not exist.

In order to see how the integrable tori change under perturbation, we study

\[
H(p, q) = H_0(p, q) + \varepsilon H_1(p, q),
\]

where \(H_0\) is an integrable Hamiltonian, \(H_1\) is non-integrable, and \(\varepsilon \in \mathbb{R}\) is small.

We know that under small perturbations some Hamiltonian systems stay close to integrable (Solar System), but on the other hand, some are globally chaotic (statistical mechanics).

Kolmogorov (1954), Arnold (1963), and Moser (1973) proved a result that for sufficiently small \(\varepsilon\), most of the tori survive the perturbation (the KAM theorem).
The actual proof is difficult [see, e.g., Arnold]. Some considerations follow. In action-angle variables of $H_0$;

$$H(J, \theta) = H_0(J) + \varepsilon H_1(J, \theta).$$

If there are tori in $H$, we have a new set of action-angle variables $(J', \theta')$ for which

$$H(J, \theta) = H'(J'),$$

and which are obtained by a generating function $S(J', \theta)$ such that

$$J = \frac{\partial S}{\partial \theta},$$
$$\theta' = \frac{\partial S}{\partial J'}.$$

The corresponding Hamilton-Jacobi equation (HJE) is

$$H \left( \frac{\partial S}{\partial \theta}, \theta \right) = H'(J').$$

We look for a solution in the form of a power series:

$$S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \ldots.$$  

We set $S_0 = J' \cdot \theta$, because $\varepsilon = 0$ then corresponds to $J = J'$, $\theta' = \theta$. Substitution to HJE gives

$$H_0 \left( J' + \varepsilon \frac{\partial S_1}{\partial \theta} + \varepsilon^2 \frac{\partial S_2}{\partial \theta} + \ldots \right) + \varepsilon H_1 \left( J' + \varepsilon \frac{\partial S_1}{\partial \theta} + \ldots, \theta \right) = H'(J').$$

By differentiating at $J'$ with respect to the first order $\varepsilon$-terms we have

$$H_0(J') + \varepsilon \frac{\partial H_0}{\partial J'} \frac{\partial S_1}{\partial \theta} + \varepsilon H_1(J', \theta) = H'(J').$$

(5)

Next, we write $\theta$-dependent terms as Fourier series;

$$H_1(J', \theta) = \sum_m H_{1,m}(J') \exp(im \cdot \theta),$$
$$S_1(J', \theta) = \sum_m S_{1,m}(J') \exp(im \cdot \theta),$$
where $H_1, S_1$ are real-valued, and the coefficients $H_{1,m}, S_{1,m} \in \mathbb{C}$, $m \in \mathbb{Z}^N$. By substituting into (5), and requiring that $H'$ is a function of $J'$ only, we have

$$S_1 = i \sum_m \frac{H_{1,m}(J')}{m \cdot \omega_0(J')} \exp(im \cdot \theta),$$

where $\omega_0(J) = \partial H_0(J)/\partial J$ are the unperturbed frequencies. Similar terms $S_2, S_3, \ldots$ follow, if this technique is applied to higher orders of $\epsilon$. Clearly, the resonant tori for which $m \cdot \omega = 0$ play a special role (the problem of small denominators). The KAM theorem essentially proves that the series for $S$ converges for ‘very nonresonant’ tori with a frequency vector $\omega$ satisfying

$$|m \cdot \omega| > K(\omega)|m|^{-[N+1]}, \quad \forall m \in \mathbb{Z}^N \setminus \{0\},$$

where $K(\omega) > 0$, and $|m| = |m_1| + |m_2| + \ldots + |m_N|$. The Lebesgue measure of the complement of this set, around resonant tori, goes to zero, as $\epsilon \to 0$. However, since the unperturbed resonant tori are dense in the phase space, whenever $\epsilon > 0$, arbitrarily near each surviving torus there is a region of destroyed resonant tori which can host irregular motion.

7 The fate of resonant tori

*Twist map* $(r_{n+1}, \phi_{n+1}) = M_0(r_n, \phi_n)$ is a model for the SOS map of an integrable Hamiltonian $H_0(J)$ with $N = 2$ [Ott, fig. 7.7];

$$r_{n+1} = r_n,$$

$$\phi_{n+1} = [\phi_n + 2\pi R(r_n)] \mod 2\pi,$$

where $R(r) = \omega_1/\omega_2$ is the ratio of frequencies for the torus $r$, and the SOS is drawn at $\theta_2 = \text{constant}$. On a resonant torus at $r = \hat{r}$, $R(\hat{r}) = j/k$, $j, k \in \mathbb{Z}$ such that $k \omega_1 - j \omega_2 = 0$, and $k$ applications of the map return to the original point;

$$M_0^k(r, \phi) = (r, (\phi + 2\pi j) \mod 2\pi) = (r, \phi).$$

Hence, each point on $r = \hat{r}$ is a fixed point of $M_0^k$. We assume that $R$ is smooth and increasing. In such a case there are circles $r_- < \hat{r} < r_+$ in such a way such that under $M_0^k$, the points on $r_-$ and $r_+$ travel clockwise and counterclockwise, respectively [Ott, fig. 7.8a].
When $H_0$ is perturbed by the term $\varepsilon H_1$, we have a slightly changed map $M_\varepsilon$:

$$r_{n+1} = r_n + \varepsilon g(r_n, \phi_n),$$
$$\phi_{n+1} = \left[\phi_n + 2\pi R(r_n) + \varepsilon h(r_n, \phi_n)\right] \mod 2\pi.$$

If $\varepsilon$ is small enough, we still have the (distorted) circles $r_-$ and $r_+$ where the points travel to opposite directions under $M^k_\varepsilon$. Thus, there must be a curve $\check{r}_\varepsilon(\phi)$ in between where $M^k_\varepsilon$ moves points only in radial direction [Ott, fig. 7.8b].

When $M^k_\varepsilon$ is applied to the points on $\check{r}_\varepsilon(\phi)$, we obtain another curve $\check{r}'_\varepsilon(\phi)$ [Ott, fig. 7.9]. Since $M^k_\varepsilon$ is volume-preserving, the two curves intersect at a finite and even number of points which are fixed points of $M^k_\varepsilon$. Since we know the directions where the points travel under $M^k_\varepsilon$ with respect to the curves, we can identify half of the fixed points as elliptic and the other half as hyperbolic. [Ott, fig. 7.10] This is the Poincaré-Birkhoff theorem.

The number of elliptic (and hyperbolic) points is equal to $k$. An elliptic orbit on the SOS circles around the elliptic points. [computer demo, logarithmic potential] The neighbourhood of an elliptic point can be further modelled by another perturbed twist mapping, resulting in a new set of elliptic and hyperbolic points in a smaller scale. If this process is continued ad infinitum, a fractal-like structure of the phase space is revealed, with subsequently smaller tori winding on bigger ones [Lichtenberg & Lieberman, fig. 3.5].

A hyperbolic point lies at an intersection of an unstable and stable manifold, $W_u$ and $W_s$, respectively. Consider a repeated SOS mapping $(M^k_\varepsilon)^n$. A point on $W_s$ approaches the hyperbolic point, as $n \to \infty$. The same happens to a point on $W_u$ as $n \to -\infty$.

The two manifolds $W_u$ and $W_s$ connect to another hyperbolic point in a way that resembles the phase portrait of a simple pendulum. [Lichtenberg & Lieberman, fig. 3.3a] For the pendulum, $W_u$ and $W_s$ between the points coincide, forming a smooth separatrix. However, in the perturbed system, the $W_u$ and $W_s$ are generally different, and intersect at a homoclinic point. This point, mapped by $(M^k_\varepsilon)^n$ must also lie on both $W_u$ and $W_s$. Hence, we conclude that there must be an infinite number of homoclinic points as $n \to \pm \infty$.

Since $M^k_\varepsilon$ is area-preserving, $W_u$ ($W_s$) must oscillate wildly as $n \to \infty$ ($n \to -\infty$). Near a hyperbolic point, these oscillations overlap, and form a
region where points from $W_u$ and $W_s$ are mixed (homoclinic tangle). [Lichtenberg & Lieberman, fig. 3.4b] Orbits in this region show extreme sensitivity to initial conditions, and hence, are chaotic [computer demo, logarithmic potential].

8 Transition to global chaos

In two degrees of freedom, the chain of resonant islands and chaotic regions surrounding them are confined between surviving KAM-tori. As the perturbation strength $\varepsilon$ increases, chaotic regions grow, and more KAM-tori are destroyed. When all KAM-tori are destroyed, an can wander anywhere in the energy surface, and the motion becomes ergodic [computer demo, standard map].

For the standard map, the rotation number of the last surviving KAM-torus is the golden ratio which is the farthest away from any rational number in the sense that its representation as a continued fraction is

$$\frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \cdots}}$$

For $N = 2$ the 2-tori can enclose volumes in the three dimensional energy surface $H(p,q) = E$. For $N > 2$ the situation is qualitatively different; e.g., for $N = 3$ the 3-tori cannot enclose 5-dimensional volumes much like lines cannot enclose 3-dimensional volumes. For any $\varepsilon > 0$, stochastic regions form a web where orbits can, in principle, reach any part of the energy surface through Arnold diffusion. However, Nekhoroshev theorem tells us that the Arnold diffusion happens very slowly; at time scales $\exp(1/\varepsilon)$ or even slower.