Maximal compatible extensions of partial orders

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Abstract. We give a complete description of maximal compatible partial orders on the mono-unary algebra \((A, f)\), where \(f : A \rightarrow A\) is an arbitrary unary operation.

1. INTRODUCTION

The well known Szpilrajn theorem ([3]) asserts that any partial order \(\leq_r\) (or \(\leq_{R}\)) on a set \(A\) can be extended to a linear order \(\leq_{R}\). As a consequence we obtain that the maximal partial orders (with respect to the containment relation) on \(A\) are exactly the linear orders of \(A\). If \(f : A \rightarrow A\) is a unary operation, then we can restrict our consideration to the so called compatible partial orders of \((A, f)\), i.e. to partial orders with the following property: \(x \leq_r y\) implies \(f(x) \leq_r f(y)\) for all \(x, y \in A\). In the present paper we shall investigate the compatible extensions of a given \(r\) in a partially ordered mono-unary algebra \((A, f, \leq_r)\). Using \(f\)-prohibited pairs, for compatible partial orders we define the notion of \(f\)-quasilinearity. Our main result states, that a compatible partial order \(r\) on \((A, f)\) can always be extended to a compatible \(f\)-quasilinear partial order \(R\). As a consequence, we obtain that the maximal compatible partial orders on \((A, f)\) are exactly the compatible \(f\)-quasilinear partial orders. It turns out, that a compatible \(f\)-quasilinear partial order is linear if and only if the function \(f\) has no proper cycle (acyclic according to the terminology of [2]). Thus the following main theorem of [2] will appear as a special case of our Theorem 4.2:

Let \(f : A \rightarrow A\) be an acyclic function (there is no \(c \in A\) such that \(f(c) \neq c\) and \(f^n(c) = c\) for some integer \(n \geq 2\)) and \(r \subseteq A \times A\) a compatible partial order on \((A, f)\). Then there exists a compatible linear order \(R \subseteq A \times A\) on \((A, f)\) with \(r \subseteq R\).

On the other hand, we shall make extensive use of the above result in proving Theorem 4.2.

2. COMPONENTS, CYCLES AND DISTANCE

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Let \( f : A \to A \) be a function (unary operation on the set \( A \)). We define the relation \( \sim_f \) as follows: for \( x, y \in A \) let \( x \sim_f y \) if \( f^k(x) = f^l(y) \) for some integers \( k \geq 0 \) and \( l \geq 0 \). It is straightforward to see that \( \sim_f \) is an equivalence on \( A \). The equivalence class \([x]_f \) of an element \( x \in A \) is called the \( f \)-component of \( x \). Clearly, \([x]_f \subseteq A \) is a subalgebra in \((A, f)\), i.e. \( f([x]_f) \subseteq [x]_f \). An element \( c \in A \) is called cyclic with respect to \( f \) (or cyclic in \((A, f)\)), if \( f^m(c) = c \) for some integer \( m \geq 1 \).

For a cyclic element

\[
n = n(c) = \min\{m \mid m \geq 1 \text{ and } f^m(c) = c\}
\]
is called the period of \( c \) (or the length of the cycle \( C = \{c, f(c), \ldots, f^{n-1}(c)\} \)), it is easy to prove that \( C \) has exactly \( n \) elements and \( f(C) = C \) and \( f^k(c) = f^l(c) \) holds if and only if \( k - l \) is divisible by \( n \). A pair \((x, y) \in A \times A\) is called \( f \)-prohibited, if we can find integers \( k \geq 0 \), \( l \geq 0 \) and \( m \geq 2 \) such that \( m \) is not a divisor of \( k - l \) and \( f^k(x), f^{k+1}(x), \ldots, f^{k+m-1}(x) \) are distinct elements, moreover \( f^{k+m}(x) = f^k(x) = f^l(y) \). We note that, for an \( f \)-prohibited pair \((x, y)\) and for an integer \( k \geq 0 \) as above, we have \( y \in [x]_f \) and \( f^k(x) \) is a cyclic element in \([x]_f \) of period \( m \). It is easy to verify, that a pair \((x, y) \in A \times A\) is \( f \)-prohibited, if and only if \( f^k(x) = f^l(y) \) is cyclic and \( f^{k+s}(x) \neq f^{k+l}(y) \) for some integers \( k \geq 0 \) and \( l \geq 0 \) (the latter condition can be replaced by \( f^t(x) \neq f^t(y) \) for all integers \( t \geq 0 \)).

The distance between an element \( y \in [x]_f \) and a given cyclic element \( c \in [x]_f \) is defined in part (1) of the following proposition.

2.1. Proposition. Let \( y \in [x]_f \) and \( c \in [x]_f \) a cyclic element of period \( n \geq 1 \). Then we have the following.

1. There exists an integer \( t \geq 0 \) such that \( f^t(y) = c \). Let \( d(y, c) = \min\{t \mid t \geq 0 \text{ and } f^t(y) = c\} \) denote the distance of \( y \) from \( c \).

2. \( d(f(c), c) = n - 1 \) and for \( y \neq c \) we have \( d(f(y), c) = d(y, c) - 1 \).

3. All cyclic elements of \([x]_f \) are in \( C = \{c, f(c), \ldots, f^{n-1}(c)\} \) and each element in \( C \) is cyclic of period \( n \).

4. If \( l \geq 0 \) is an integer, then \( f^l(y) = c \) holds if and only if \( l \geq d(y, c) \) and \( l - d(y, c) \) is divisible by \( n \).

5. \((x, y) \) is \( f \)-prohibited if and only if \( n \geq 2 \) and \( d(x, c) - d(y, c) \) is not divisible by \( n \).

Proof.

1. \( y \in [x]_f \) and \( c \in [x]_f \) imply that \( c \sim_f y \), so we have \( f^k(c) = f^l(y) \) for some integers \( k \geq 0 \) and \( l \geq 0 \). Now

\[
f^{nk-k+1}(y) = f^{nk-k}(f^k(y)) = f^{nk-k}(f^k(c)) = f^{nk}(c) = c.
\]

2. \( d(f(c), c) = n - 1 \) follows from the facts, that \( c, f(c), \ldots, f^{n-1}(c) \) are distinct elements and \( f^{n-1}(f(c)) = f^n(c) = c \). If \( y \neq c \), then \( d = d(y, c) \geq 1 \), \( f^{d-1}(f(y)) = f^d(y) = c \) and \( c \notin \{y, f(y), f^2(y), \ldots, f^{d-1}(y)\} \) ensure that \( d(f(y), c) = d - 1 \).

3. For a cyclic element \( c' \in [x]_f \), the application of part (1) ensures the existence of an integer \( t \geq 0 \) with \( f^t(c) = c' \). As we have already noted \( f^t(c) \in C \), the rest of part (3) is obvious.
(4) If $0 \leq l \leq d(y, c) - 1$, then $f^l(y) \neq c$ by our definition of the distance. If $l - d(y, c) \geq 0$, then
\[ f^l(y) = f^{l-d(y,c)}(f^{d(y,c)}(y)) = f^{l-d(y,c)}(c). \]
Since $c$ is cyclic of period $n$, the equality $f^{l-d(y,c)}(c) = c$ holds if and only if $l - d(y, c)$ is divisible by $n$.

(5) If $(x, y)$ is $f$-prohibited, then $f^k(x), f^{k+1}(x), \ldots, f^{k+m-1}(x)$ are distinct elements, $f^{k+m}(x) = f^k(x) = f^l(y)$ for some integers $k \geq 0$, $l \geq 0$ and $m \geq 2$ with $m \mid k-l$. Now $f^k(x) \in [x]_f$ is a cyclic element of period $m$, the use of part (3) gives that $f^k(x) \in C$ and $m = n$. Take $z = f^k(x) = f^l(y)$, then $f^{d(z,c)}(z) = c$ implies
\[ f^{d(z,c)+k}(x) = f^{d(z,c)}(f^k(x)) = c = f^{d(z,c)}(f^l(y)) = f^{d(z,c)+l}(y), \]
whence $n \mid (d(z,c) + k) - d(x, c)$ and $n \mid (d(z,c) + l) - d(y, c)$ can be obtained by part (4). Thus we have
\[ n \mid (k - l) - (d(x, c) - d(y, c)) \]
in addition to $n \mid k - l$. It follows that $n \nmid d(x, c) - d(y, c)$. If $n \nmid d(x, c) - d(y, c)$ then
\[ f^{d(z,c)}(x) = c, f^{d(z,c)+1}(x) = f(c), \ldots, f^{d(z,c)+n-1}(x) = f^{n-1}(c) \]
are distinct elements and
\[ f^{d(z,c)+n}(x) = f^n(c) = c = f^{d(z,c)}(x) = f^{d(y,c)}(y). \]
In view of the above facts, we deduce that $(x, y)$ is an $f$-prohibited pair.$\square$

2.2. Proposition. If $(A, f, \leq_r)$ is a partially ordered mono-unary algebra, then we have the following.

(1) If $c \in A$ is a cyclic element of period $n \geq 1$, then $C = \{c, f(c), \ldots, f^{n-1}(c)\}$ is an antichain with respect to $\leq_r$: for $0 \leq i < j \leq n - 1$ the elements $f^i(c)$ and $f^j(c)$ are incomparable with respect to $\leq_r$.

(2) If $(x, y) \in A \times A$ is an $f$-prohibited pair, then $(x, y) \notin r$ and $(y, x) \notin r$, i.e. $x$ and $y$ are incompatible elements with respect to $\leq_r$.

Proof.

(1) Take $c^* = f^l(c)$ and $t = j - i$, then $f^t(c^*) = f^l(c)$. Now $c^* \leq_r f^t(c^*)$ would imply
\[ c^* \leq_r f^t(c^*) \leq_r f^{2t}(c^*) \leq_r \ldots \leq_r f^{nt}(c^*) = c^* \]
in contradiction with $c^* \neq f^t(c^*)$. The reverse relation $f^t(c^*) \leq_r c^*$ would lead to a similar contradiction.

(2) Let $f^k(x), f^{k+1}(x), \ldots, f^{k+m-1}(x)$ be distinct elements and $f^{k+m}(x) = f^k(x) = f^l(y)$ for some integers $k \geq 0$, $l \geq 0$ and $m \geq 2$ with $m \mid k-l$. The assumption $x \leq_r y$ would imply
\[ f^{k+l}(x) \leq_r f^{k+l}(y) \]
for the elements $f^{k+l}(x)$ and $f^{k+l}(y) = f^k(f^l(y)) = f^k(f^k(x)) = f^{2k}(x)$ of the cycle
\[ C = \{f^k(x), f^{k+1}(x), \ldots, f^{k+m-1}(x)\}. \]
Using \( m \nmid k - l \), for the cyclic element \( c = f^k(x) \) of period \( m \), we obtain that \( f^l(c) \neq f^k(c) \) i.e. that \( f^{k+l}(x) \neq f^{2k}(x) \). Now we are in contradiction with part (1). The case \( y \leq_r x \) can be treated similarly. 

3. THE ORDER COMPONENTS OF \((A, f, \leq_r)\)

Let \((A, f, \leq_r)\) be a partially ordered mono-unary algebra and consider the factor set 

\[ B = A/\sim_f = \{ [x]_f \mid x \in A \}. \]

We define the relation \( \prec_r \) on \( B \) as follows: for \( x, y \in A \) let \( [x]_f \prec_r [y]_f \) if \( x \leq_r y \) for some \( x_1 \in [x]_f \) and \( y_1 \in [y]_f \).

3.1. Proposition.

(1) \( \prec_r \) is a quasiorder (reflexive and transitive) on \( B \).

(2) If \([x]_f \prec_r [y]_f\) and \([y]_f \prec_r [z]_f\) for the \( f \)-components \([x]_f \neq [y]_f\), then there is no cyclic element \( c \in [x]_f \cup [y]_f \) of period \( n \geq 1 \).

Proof.

(1) In order to see the transitivity of \( \prec_r \), let \([x]_f \prec_r [y]_f\) and \([y]_f \prec_r [z]_f\). Now we have \( x_1 \leq_r y_1 \) and \( y_1 \leq_r z_1 \) for some \( x_1 \in [x]_f \), \( y_1, y'_1 \in [y]_f \) and \( z_1 \in [z]_f \). Since \( y_1 \sim_r y'_1 \), we can find integers \( k \geq 0 \) and \( l \geq 0 \) such that \( f^k(y_1) = f^l(y'_1) \). In view of 

\[ f^k(x_1) \leq_r f^k(y_1) = f^l(y'_1) \leq_r f^l(z_1), \]

the relations \( f^k(x_1) \in [x]_f \) and \( f^l(z_1) \in [z]_f \) imply that \([x]_f \prec_r [z]_f\).

(2) Suppose that \( c \in [x]_f \) is a cyclic element of period \( n \geq 1 \). The relations \([x]_f \prec_r [y]_f\) and \([y]_f \prec_r [x]_f\) ensure the existence of elements \( x_1, x_2 \in [x]_f \) and \( y_1, y_2 \in [y]_f \) with the properties \( x_1 \leq_r y_1 \) and \( y_2 \leq_r x_2 \). Using part (1) of Proposition 2.1, we obtain that 

\[ f^{t_1}(x_1) = c = f^{t_2}(x_2) \]

for some integers \( t_1 \geq 0 \) and \( t_2 \geq 0 \). Since \( f^{t_1}(y_1) \sim f^{t_2}(y_2) \), we can find integers \( k \geq 0 \) and \( l \geq 0 \) such that 

\[ f^k(f^{t_1}(y_1)) = f^l(f^{t_2}(y_2)). \]

The compatibility of \( \leq_r \) gives the following relations

\[ f^k(c) = f^k(f^{t_1}(x_1)) \leq_r f^k(f^{t_1}(y_1)) = f^l(f^{t_2}(y_2)) \leq_r f^l(f^{t_2}(x_2)) = f^l(c), \]

where \( f^k(c) \) and \( f^l(c) \) are cyclic elements. On applying part (1) of Proposition 2.2, we obtain that \( f^k(c) = f^k(f^{t_1}(y_1)) = f^l(c) \) in contradiction with \([x]_f \cap [y]_f = \emptyset\). 

The relation \( \equiv_r \) is defined on \( B = A/\sim_f \) as follows: for \( x, y \in A \) let \([x]_f \equiv_r [y]_f \) if \([x]_f \prec_r [y]_f\) and \([y]_f \prec_r [x]_f\). It is well known, that starting from the quasiorder \( \prec_r \), the above definition provides an equivalence on \( B \). We define the order component of \( x \) in \((A, f, \leq_r)\) by 

\[ \langle x \rangle = \bigcup_{y \in A \text{ and } [y]_f \equiv_r [x]_f} [y]_f. \]
Clearly, \([x]_f \subseteq \langle x \rangle \subseteq A\) and \(\langle x \rangle\) is a subalgebra in \((A, f)\), which corresponds to the \(\equiv_r\) equivalence class \([x]_f\) of \([x]_f\) in \(B\). It is easy to see that \(\{\langle x \rangle : x \in A\}\) is a partition of \(A\):

\[
\bigcup_{x \in A} \langle x \rangle = A \quad \text{and} \quad \langle x \rangle = \langle y \rangle \quad \text{or} \quad (\langle x \rangle \cap \langle y \rangle) = \emptyset \quad \text{for all} \quad x, y \in A.
\]

If \(c \in \langle x \rangle\) is a cyclic element, then part (2) of Proposition 3.1 gives that \(\langle x \rangle = [x]_f\).

We shall make use of the partial order \(\prec_r\) on \(B/\equiv_r\), which can be derived from \(\prec_r\) in a natural way: \(\langle x \rangle \prec_r \langle y \rangle\) if \([x]_f \prec_r [y]_f\).

3.2. Lemma. Let \((A, f, \leq_r)\) be a partially ordered mono-unary algebra. If \(x \in A\) and there is no cyclic element in \(\langle x \rangle\), then there exists a linear order \(\rho\) on \(\langle x \rangle\) with the following properties:

\begin{enumerate}
\item \(\rho\) is compatible on \((\langle x \rangle, f)\): \((u, v) \in \rho \Rightarrow (f(u), f(v)) \in \rho\) for all \(u, v \in \langle x \rangle\),
\item \(\rho\) is an extension of \(\leq_r\) on the elements of \(\langle x \rangle\): \(\rho \cap (\langle x \rangle \times \langle x \rangle) \subseteq \rho\).
\end{enumerate}

**Proof.** The absence of cyclic elements ensures that \(f : \langle x \rangle \rightarrow \langle x \rangle\) is a so called acyclic function preserving the partial order \(\rho \cap (\langle x \rangle \times \langle x \rangle)\). A straightforward application of the Main Theorem in [2] gives the existence of the desired \(\rho\). \(\square\)

3.3. Lemma. Let \((A, f, \leq_r)\) be a partially ordered mono-unary algebra. If \(x \in A\) and \(c \in \langle x \rangle\) is a cyclic element of period \(n \geq 1\), then there exists a partial order \(\rho\) on \(\langle x \rangle = [x]_f\) with the following properties:

\begin{enumerate}
\item \(\rho\) is compatible on \((\langle x \rangle, f)\): \((u, v) \in \rho \Rightarrow (f(u), f(v)) \in \rho\) for all \(u, v \in [x]_f\),
\item \(\rho\) is an extension of \(\leq_r\) on the elements of \([x]_f\): \(\rho \cap ([x]_f \times [x]_f) \subseteq \rho\),
\item \([x]_f = E_0 \cup E_1 \cup \ldots \cup E_{n-1}\) is a pairwise disjoint union, where each set \(E_i = \{u \in [x]_f : d(u, c) - i \text{ is divisible by } n\} \land \ 0 \leq i \leq n - 1\)
\end{enumerate}

is a chain with respect to \(\rho\), and for \(i \neq j\) the elements of \(E_i \times E_j\) are \(f\)-prohibited pairs.

**Proof.** Let \(E = [x]_f\) and consider the equivalence relation \(\varepsilon = \Delta_E \cup (C \times C)\) on \(E\), where \(\Delta_E\) is the diagonal of \(E \times E\) and \(C = \{c, f(c), \ldots, f^{n-1}(c)\}\) is the set of cyclic elements in \(E\). Clearly, \([u]_\varepsilon = [u]\) if \(u \in E \setminus C\) and \([u]_\varepsilon = C\) if \(u \in C\). Using the factor set \(E^* = E/\varepsilon\), define a function \(f^* : E^* \rightarrow E^*\) and a relation \(r^* \subseteq E^* \times E^*\) as follows: \(f^*([u]_\varepsilon) = [f(u)]_\varepsilon\) and \(r^* = \pi\) is the transitive closure of the reflexive relation

\[
s = \{(u, v) \in E \times E : u \leq_r v\}\.
\]

The implication \(u' \in [u]_\varepsilon \Rightarrow f(u') \in [f(u)]_\varepsilon\) follows from \(f(C) \subseteq C\), thus the definition of \(f^*\) is correct and for \([u]_\varepsilon, [v]_\varepsilon\) \(\in s\) we have

\[
(f^*([u]_\varepsilon), f^*([v]_\varepsilon)) = ([f(u)]_\varepsilon, [f(v)]_\varepsilon)
\]

with \(u' \leq_r v'\) for some \(u' \in [u]_\varepsilon\) and \(v' \in [v]_\varepsilon\). Since \(f(u') \leq_r f(v')\) for \(f(u') \in [f(u)]_\varepsilon\) and \(f(v') \in [f(v)]_\varepsilon\), we obtain that \((f^*([u]_\varepsilon), f^*([v]_\varepsilon)) \in s\), i.e. that \(f^*\) preserves the relation \(s\). As a consequence, we also obtain that \(f^*\) preserves the transitive closure \(r^* = \pi\) of \(s\). We claim, that \(r^*\) is a partial order on \(E^*\). It is
enough to show, that there is no proper cycle in \( E^* \) with respect to \( s \). If a proper cycle
\[
[u_1]_e s[u_2]_e s \ldots s[u_k]_e s[u_1]_e
\]
does not contain \( C \), then we have
\[
u_1 \leq_r u_2 \leq_r \ldots \leq_r u_k \leq_r u_1
\]
implying that \( u_1 = u_2 = \ldots = u_k \), a contradiction. If \( C \) appears in a proper cycle, then we can exhibit a segment of it as
\[
C s[v_1]_e s[v_2]_e s \ldots s[v_k]_e s C,
\]
where \( v_1, v_2, \ldots, v_k \notin C \). Now we have
\[
c' \leq_r v_1 \leq_r v_2 \leq_r \ldots \leq_r v_k \leq_r c''
\]
for some \( c', c'' \in C \). Applying part (1) of Proposition 2.2 gives that \( c' = c'' \). Thus the elements \( v_1 = v_2 = \ldots = v_k = c' = c'' \) are in \( C \), a contradiction. The only cyclic element of \( (E^*, f^*) \) is \( C \) and \( f^*(C) = C \), so we can apply the Main Theorem of [2] to the partially ordered algebra \( (E^*, f^*, r^*) \), in order to get a compatible linear order \( \rho^* \) on \( (E^*, f^*) \) with \( r^* \subseteq \rho^* \). We claim, that
\[
\rho = \{ (u, v) \mid u, v \in E, ([u]_e, [v]_e) \in \rho^* \text{ and } n \mid d(u, c) - d(v, c) \}
\]
is one of the desired relations on \( E \).

The reflexive and transitive properties of \( \rho \) can be easily verified. Let \( (u, v) \in \rho \) and \( (v, u) \in \rho \), then \( ([u]_e, [v]_e) \in \rho^* \) and \( ([v]_e, [u]_e) \in \rho^* \) imply \( [u]_e = [v]_e \), whence \( u = v \) or \( u, v \in C \) can be derived. If \( u, v \in C \), then we also have \( u = v \) by \( n \mid d(u, c) - d(v, c) \) and the compatibility of \( \rho^* \).

We proved the antisymmetric property of \( \rho \).

If \( (u, v) \in \rho \), then \( ([u]_e, [v]_e) \in \rho^* \) and the compatibility of \( \rho^* \) provides that
\[
([f(u)]_e, [f(v)]_e) = (f^*([u]_e), f^*([v]_e)) \in \rho^*.
\]
Using part (2) of Proposition 2.1, we obtain \( n \mid d(f(u), c) - d(f(v), c) \) as a consequence of the divisibility \( n \mid d(u, c) - d(v, c) \). We proved that \( (f(u), f(v)) \in \rho \).

If \( u \leq_r v \), then first we get \( ([u]_e, [v]_e) \in s \) and next \( ([u]_e, [v]_e) \in \rho^* \subseteq \rho^* \). For \( n \geq 2 \) the relation \( n \mid d(u, c) - d(v, c) \) would imply that \( (u, v) \) is an \( f \)-prohibited pair in \( E \) (see part (5) in Proposition 2.1). According to part (2) of Proposition 2.2, this contradicts \( u \leq_r v \). Thus we have \( n \mid d(u, c) - d(v, c) \) and \( (u, v) \in \rho \), proving \( r \subseteq \rho \).

For \( u, v \in E_i \) the divisibility \( n \mid d(u, c) - d(v, c) \) follows from \( n \mid d(u, c) - i \) and \( n \mid d(v, c) - i \). Since \( \rho^* \) is linear, either \( ([u]_e, [v]_e) \in \rho^* \) or \( ([v]_e, [u]_e) \in \rho^* \) holds. Thus we have either \( (u, v) \in \rho \) or \( (v, u) \in \rho \), proving that \( E_i \) is a chain with respect to \( \rho \).

If \( i \neq j \) and \( (u, v) \in E_i \times E_j \), then \( n \mid d(u, c) - i \) and \( n \mid d(v, c) - j \) imply that \( d(u, c) - d(v, c) \) is not divisible by \( n \). Thus part (5) of Proposition 2.1 ensures that \( (u, v) \) is an \( f \)-prohibited pair. □

**3.4. Remark.** According to Proposition 3.6 in [1], the convexity of the antichain \( C \) implies that \( e = \Delta_E \cup (C \times C) \) is an order congruence of \( (E, f, r \cap (E \times E)) \).

4. THE MAIN RESULTS

A compatible partial order \( R \) on a mono-unary algebra \((A, f)\) is called \( f \)-quasilinear,
if \((x, y) \in R \) or \((y, x) \in R \) for all non \( f \)-prohibited pairs \((x, y) \in A \times A \). In view of part (2) of Proposition 2.2, we have the following simple observation.
4.1. Proposition. If a compatible partial order \( R \) on a mono-unary algebra \((A, f)\) is \( f \)-quasilinear, then it is maximal (with respect to containment) among the compatible partial orders of \((A, f)\).

4.2. Theorem. If \((A, f, \leq_r)\) is a partially ordered mono-unary algebra, then there exists a compatible partial order \( R \) on \((A, f)\) with the following properties:

1. \( R \) is an extension of \( r \), i.e. \( r \subseteq R \).
2. \( R \) is \( f \)-quasilinear.

Proof. Let \( \preceq_\lambda \) be an arbitrary linear extension of the partial order \( \preceq_r \) on the set \( B/ \equiv_r \) of order components in \((A, f, \leq_r)\), where \( B = A/ \sim_f \). If \( x \in A \) and there is no cyclic element in \( \langle x \rangle \), then fix a compatible linear order \( \rho(x) \) on \( \langle x \rangle \) with the properties described in Lemma 3.2. If there is a cyclic element of period \( n \geq 1 \) in \( \langle x \rangle \), then fix a compatible partial order \( \rho(x) \) on \( \langle x \rangle = [x]_f \) with the properties described in Lemma 3.3. We claim, that

\[
R = \{ (x, y) \in A \times A \mid \langle x \rangle \preceq_\lambda \langle y \rangle \text{ and } (x, y) \in \rho(x) \text{ in case of } \langle x \rangle = \langle y \rangle \}
\]

is one of the desired relations on \( A \).

The reflexive, antisymmetric and transitive properties of \( R \) can be easily verified. In order to prove the compatibility of \( R \), it is enough to note that \( (f(x)) = \langle x \rangle \) and that \( \rho(x) \) is a compatible partial order on \( \langle x, f \rangle \).

If \( x \leq_r y \) then \( [x]_f \preceq_r [y]_f \), whence we obtain \( \langle x \rangle \preceq_r \langle y \rangle \) as well as \( \langle x \rangle \preceq_\lambda \langle y \rangle \). In case of \( \langle x \rangle = \langle y \rangle \) the relation \( (x, y) \in \rho(x) \) follows from \( r \cap (\langle x \rangle \times \langle x \rangle) \subseteq \rho(x) \). Thus we have \( (x, y) \in R \), proving \( r \subseteq R \).

If \( x, y \in A \) are incomparable elements with respect to \( R \), then the linearity of \( \preceq_\lambda \) implies that \( \langle x \rangle = \langle y \rangle \), whence we get \( (x, y) \notin \rho(x) \) and \( (y, x) \notin \rho(x) \). Since \( \rho(x) \) is not linear, the order component \( \langle x \rangle \) must contain a cyclic element \( c \) of period \( n \geq 2 \). In view of the properties of \( \rho(x) \) described in Lemma 3.3, we obtain that \( x \in E_i \) and \( y \in E_j \) for some \( i, j \in \{0, 1, \ldots, n-1\} \) with \( i \neq j \). Now the last property of the \( E_i \)'s guarantees that \( (x, y) \) is an \( f \)-prohibited pair. \( \square \)

4.3. Corollary. A compatible partial order \( R \) on \((A, f)\) is maximal (with respect to containment) if and only if \( R \) is \( f \)-quasilinear.

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