Labeled posets are universal

Erkko Lehtonen\textsuperscript{*}

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Abstract

Partially ordered sets labeled with \(k\) labels (\(k\)-posets) and their homomorphisms are examined. The homomorphicity order of \(k\)-posets is shown to be a distributive lattice. Homomorphicity orders of \(k\)-posets and \(k\)-lattices are shown to be universal in the sense that every countable poset can be embedded into them. Labeled posets are represented by directed graphs, and a categorical isomorphism between \(k\)-posets and their digraph representations is established.

1 Introduction

A \(k\)-labeled partially ordered set (\(k\)-poset) is an object \((P, c)\), where \(P\) is a partially ordered set and \(c\) is a function that assigns to each element of \(P\) a label from the set \([0, 1, \ldots, k - 1]\). A homomorphism between \(k\)-posets is a mapping \(h : (P, c) \to (P', c')\) that preserves both order and labels. We define the homomorphicity quasiorder on the set of all finite \(k\)-posets as follows: \((P, c) \leq (P', c')\) if and only if there is a homomorphism of \((P, c)\) to \((P', c')\).

Labeled posets have been used to represent parallel processes; see Pratt [12]. Labeled posets can be viewed as a generalization of strings. Algebraic properties of labeled posets (or partial words) have been studied by Grabowski [4], Gischer [3], Bloom and Ésik [1], and Rensink [13]. Homomorphisms of \(k\)-posets were studied in the context of Boolean hierarchies of partitions by Kosub [9], Kosub and Wagner [10], and Selivanov [14]. Kosub and Wagner were mostly concerned with \(k\)-lattices, whereas Selivanov studied \(k\)-forests. Kosub and Wagner [10] constructed an infinite descending chain and an infinite antichain in the homomorphicity order of \(k\)-lattices, and we applied this result in our analysis of certain relations between operations on a finite set that are based on composition of functions from inside with monotone functions [11]. We now try to give a systematical account of homomorphisms of \(k\)-posets.

This paper is organized as follows. In Section 2, we give the basic definitions and introduce the homomorphicity order of \(k\)-posets. We introduce the notions of a minimal \(k\)-poset and a core, and we show that these notions are in fact equivalent. The minimal \(k\)-posets, or cores, can be chosen as natural representatives of the equivalence classes of the homomorphicity order of \(k\)-posets.

\textsuperscript{*}Institute of Mathematics, Tampere University of Technology, P.O. Box 553, FI-33101 Tampere, Finland. E-mail: \texttt{erkko.lehtonen@tut.fi}. Some of the work for this paper was done while the author was visiting RUTCOR, Rutgers University.
In Section 3, we analyze the structure of the homomorphicity order of \( k \)-posets. The first main result of this paper is that this order is a distributive lattice. In Section 4, we present our second main result: the homomorphicity order of \( k \)-posets for \( k \geq 2 \) and that of \( k \)-lattices for \( k \geq 3 \) are universal in the sense that every countable poset can be embedded into them.

In Section 5, we represent \( k \)-posets as directed graphs and establish a categorical isomorphism between \( k \)-posets and their digraph representations.

Finally, in Section 6, we make some concluding remarks and point out some implications of the current results to our earlier work.

2 Labeled posets and homomorphisms

For a positive natural number \( k = \{0, 1, \ldots, k-1\} \), a \( k \)-labeled partially ordered set (\( k \)-poset) is an object \((P, \leq, c)\), where \((P, \leq)\) is a partially ordered set and \( c : P \rightarrow k \) is a labeling function. A labeled poset is a \( k \)-poset for some \( k \). Every subset \( P' \) of a \( k \)-poset \((P, \leq, c)\) may be considered as a \( k \)-poset \((P', \leq_{P'}, c_{P'})\), called a \( k \)-subposet of \((P, \leq, c)\). We often simplify these notations and write \((P, c)\) or \( P \) instead of \((P, \leq, c)\), and we simply write \( c \) for any restriction of \( c \).

If the underlying poset of a \( k \)-poset is a lattice, chain, tree, or forest, then we refer to \( k \)-lattices, \( k \)-chains, \( k \)-trees, \( k \)-forests, and so on. We denote by \( \mathcal{P}_k \) and \( \mathcal{L}_k \) the classes of all finite \( k \)-posets and \( k \)-lattices, respectively. For \( k \leq l \), every \( k \)-poset is also an \( l \)-poset. Finite \( k \)-posets can be represented by Hasse diagrams with numbers designating the labels assigned to each element; see the various figures in this paper. For general background on partially ordered sets, see any textbook on the subject, e.g., [2].

We will adopt much of the terminology used for graphs and their homomorphisms (see [6]). Let \((P, c)\) and \((P', c')\) be \( k \)-posets. An order-preserving mapping \( f : P \rightarrow P' \) such that \( c = c' \circ f \) is called a homomorphism of \((P, c)\) to \((P', c')\) and denoted \( f : (P, c) \rightarrow (P', c') \). The composition of homomorphisms is again a homomorphism.

**Lemma 2.1.** Let \((P, c)\) be a \( k \)-poset. Then a \( k \)-poset \((P', c')\) is homomorphic to \((P, c)\) if and only if any \( k \)-poset homomorphic to \((P', c')\) is also homomorphic to \((P, c)\).

**Proof.** The forward implication is clear, because the composition of homomorphisms is a homomorphism. The converse implication is also immediate, because \((P', c')\) is homomorphic to itself by the identity mapping, and hence \((P', c')\) is homomorphic to \((P, c)\) by the assumption. \( \square \)

We define a quasiorder \( \leq \) on \( \mathcal{P}_k \) as follows: \((P, c) \leq (P', c')\) if and only if there is a homomorphism of \((P, c)\) to \((P', c')\). Denote by \( \equiv \) the equivalence relation on \( \mathcal{P}_k \) induced by \( \leq \). If \((P, c) \equiv (P', c')\), we say that \((P, c)\) and \((P', c')\) are homomorphically equivalent. We denote by \( \mathcal{P}_k' \) the quotient set \( \mathcal{P}_k/\equiv \), and the partial order on \( \mathcal{P}_k' \) induced by the homomorphicity quasiorder \( \leq \) is also denoted by \( \leq \). Similarly, we denote by \( \mathcal{L}_k' \) the quotient set \( \mathcal{L}/\equiv \).

An endomorphism of \((P, c)\) is a homomorphism \( h : (P, c) \rightarrow (P, c) \). A bijective homomorphism of \((P, c)\) to \((P', c')\) whose inverse is a homomorphism of \((P', c')\) to \((P, c)\) is called an isomorphism, and \((P, c)\) and \((P', c')\) are said to
be isomorphic. Isomorphic k-posets are homomorphically equivalent by definition. Homomorphic equivalence does not imply isomorphicity; for an example of homomorphically equivalent k-posets which are not isomorphic, see Figure 1. However, as we will see later, minimal k-posets are homomorphically equivalent if and only if they are isomorphic.

A k-poset is minimal, if it is not homomorphically equivalent to any k-poset of smaller cardinality. Every finite k-poset is homomorphically equivalent to a minimal k-poset. Thus, we may choose homomorphically non-equivalent minimal k-posets as the representatives of the equivalence classes, and the partial order of \( P_k \) coincides with the restriction of \( \leq \) to these representatives in \( P_k \).

The following lemma is a straightforward generalization of one of Selivanov’s [14] basic lemmas.

**Lemma 2.2.** For any k-poset \((P, c)\), the following are equivalent.

(i) \((P, c)\) is minimal.

(ii) Every endomorphism of \((P, c)\) is injective.

(iii) \((P, c) \not\leq (P', c)\) for any proper subset \(P' \subset P\).

**Proof.** (i) \(\Rightarrow\) (iii). Suppose, on the contrary, that \((P, c) \leq (P', c)\). Then \((P', c) \leq (P, c)\) (we may take the identity embedding as the homomorphism of \((P', c)\) to \((P, c)\)). Hence \((P, c) \equiv (P', c)\) and \(|P'| < |P|\), a contradiction.

(iii) \(\Rightarrow\) (ii). Suppose that there is a non-injective endomorphism \(f: (P, c) \rightarrow (P, c)\). Letting \(P' = f(P)\), we reach a contradiction.

(ii) \(\Rightarrow\) (i). Suppose, on the contrary, that \((P, c) \equiv (Q, d)\) for a k-poset \((Q, d)\) with \(|Q| < |P|\). Then there exist homomorphisms \(f: (P, c) \rightarrow (Q, d)\) and \(g: (Q, d) \rightarrow (P, c)\). Then \(g \circ f\) is a non-injective endomorphism of \((P, c)\), a contradiction. \(\square\)

Selivanov showed that a k-forest \((P, c)\) is minimal if and only if the identity function is the only endomorphism of \((P, c)\). This is no longer true for k-posets in general, in fact, it does not even hold for k-lattices. Consider, for example, the 3-lattice of Figure 2. While being minimal, the given 3-lattice has a nontrivial endomorphism: on each horizontal row, swap the elements with similar label.

Two elements \(a\) and \(b\) of a poset \(P\) are connected, if there exists a sequence \(a_1, \ldots, a_n\) of elements of \(P\) such that \(a_1 = a, a_n = b\), and for all \(1 \leq i \leq n - 1\) either \(a_i \leq a_{i+1}\) or \(a_{i+1} \leq a_i\). A poset is connected if all pairs of its elements are connected. A connected component of a poset \(P\) is a subposet \(C \subseteq P\) that is connected and such that for every \(x \in P \setminus C\) the subposet \(C \cup \{x\}\) is not connected. It is easy to verify that any homomorphic image of a connected k-poset is connected.

**Lemma 2.3.** (i) The empty k-poset \((\emptyset, \emptyset)\) is minimal.

(ii) All singleton k-posets are minimal.
Lemma 2.4. A k-poset is minimal if and only if all its connected components are minimal and pairwise incomparable under $\leq$.

Proof. Statements (i) and (ii) are obvious.

For (iii), assume that $(P, c)$ is minimal. Suppose that a connected component $(C, c)$ of $(P, c)$ is not minimal. By Lemma 2.2, there is a non-injective automorphism $g$ of $(C, c)$. Define $f$ as

$$f(x) = \begin{cases} g(x), & \text{if } x \in C, \\ x, & \text{if } x \notin C. \end{cases}$$

Then $f$ is a non-injective endomorphism of $(P, c)$, a contradiction.

Suppose that there are distinct connected components $(C, c)$ and $(C', c)$ of $(P, c)$ with $(C, c) \leq (C', c)$, and let $g : (C, c) \to (C', c)$ be the corresponding homomorphism. Defining $f$ as above, we have a non-injective endomorphism of $(P, c)$, again a contradiction.

Assume then that the connected components of $(P, c)$ are minimal and pairwise incomparable. Again, it is easy to show that every endomorphism of $(P, c)$ is injective. By Lemma 2.2, $(P, c)$ is minimal.

A k-poset $(Q, d)$ is a retract of a k-poset $(P, c)$, if there are homomorphisms $f : (Q, d) \to (P, c)$ and $g : (P, c) \to (Q, d)$ such that $g \circ f$ is the identity map on $Q$. The homomorphism $g$ is called a retraction map and $f$ is called a coretraction map. We also say that a k-subposet $(P', c)$ of a k-poset $(P, c)$ is a retract of $(P, c)$ if there is a homomorphism $g : (P, c) \to (P', c)$ whose restriction to $P'$ equals the identity map on $P'$. The latter definition may be regarded as a special case of the former: the inclusion map $P' \to P$ may be chosen as the coretraction. A k-poset is homomorphically equivalent to all its retracts. The composition of retractions is again a retraction. A retract $(Q, d)$ of $(P, c)$ is proper if $|Q| < |P|$. A k-poset is a core if it does not have proper retracts.

Lemma 2.4. A k-poset is minimal if and only if it is a core.

Proof. Since a k-poset is homomorphically equivalent to all its retracts, it is clear that a minimal k-poset has no proper retracts and hence it is a core.

Let $(P, c)$ be a core, and suppose, on the contrary, that $(P, c)$ is not minimal. Then, by Lemma 2.2, there is proper subset $P' \subset P$ and a homomorphism $g : (P, c) \to (P', c)$. For $i \geq 1$, denote $G_i = \text{Im} g^i$. It is clear that $G_{i+1} \subseteq G_i$. Since $P$ is finite, there is an $n$ such that $G_n = G_{n-1}$. Then $g|_{G_n}$ is a permutation.
of the finite set $G_n$, and hence there is an $m \geq 1$ such that $(g|_{G_n})^m = (g^n)|_{G_n}$ equals the identity mapping on $G_n$. Thus, $g^n$ is a retraction of $(P, c)$ to the proper $k$-subposet $(G_n, c)$, a contradiction. □

**Lemma 2.5.** Homomorphically equivalent cores are isomorphic.

*Proof.* Let $(P, c)$ and $(Q, d)$ be cores, and let $f : (Q, d) \to (P, c)$ and $g : (P, c) \to (Q, d)$ be homomorphisms. By Lemma 2.4, $(P, c)$ and $(Q, d)$ are minimal, and hence $|P| = |Q|$. By Lemma 2.2, the endomorphism $f \circ g$ of $(P, c)$ is injective and hence it is a permutation of $P$. In fact, $f$ and $g$ are bijections. Then there is an $n$ such that $(f \circ g)^n$ equals the identity mapping on $P$. Hence, the inverse of $g$ is $(f \circ g)^{n-1} \circ f$ and it is a homomorphism. We conclude that $(P, c)$ and $(Q, d)$ are isomorphic. □

## 3 The structure of the homomorphicity order

First of all, we note that 1-posets simply correspond to ordinary posets. All constant maps between 1-posets are homomorphisms, and hence all nonempty 1-posets are homomorphically equivalent. The empty 1-poset $(\emptyset, \emptyset)$ is homomorphic to every 1-poset. Thus, the homomorphicity order $(\mathcal{P}_1', \leq)$ of 1-posets is just the two-element chain $\{(\emptyset, \emptyset)\} < \mathcal{P}_1' \setminus \{(\emptyset, \emptyset)\}$. From now on, we assume that $k \geq 2$.

We introduce two operations of $k$-posets: the disjoint union and the label-matching product.

The *disjoint union* of a finite family $\{(P_i, c_i)\}_{i \in I}$ of pairwise disjoint $k$-posets is the $k$-poset

$$\bigcup_{i \in I} (P_i, c_i) = \left( \bigcup_{i \in I} P_i, \bigcup_{i \in I} c_i \right),$$

where the order $\leq$ on $\bigcup_{i \in I} P_i$ is defined as $x \leq y$ in $\bigcup_{i \in I} P_i$ if and only if $x \leq y$ in $P_i$ for some $i \in I$. We denote the disjoint union of two $k$-posets $(A, c)$ and $(B, d)$ by $(A, c) \cup (B, d)$ or simply by $A \cup B$.

The *label-matching product* of $(A, c)$ and $(B, d)$, denoted $(A, c) \odot (B, d)$ or simply $A \otimes B$, is the $k$-poset $(P, f)$, where

$$P = \{(x, y) \in A \times B : c(x) = d(y)\},$$

$(x, y) \leq (x', y')$ in $A \otimes B$ if and only if $x \leq x'$ in $A$, $y \leq y'$ in $B$, $c(x) = d(x)$, and $c(y) = d(y)$; and the labeling $f$ is defined as $f(x, y) = c(x) = d(y)$.

See Figure 3 for an illustrative example of the label-matching product of $k$-posets. For the sake of clarity, we show both the names of elements and the labels in the Hasse diagrams. Note that this example shows that the label-matching product of $k$-lattices is not necessarily a $k$-lattice and also that the label-matching product of minimal $k$-posets is not necessarily minimal.

**Proposition 3.1.** Let $(P, c)$, $(P', c')$ be cores. Then the equivalence classes of $P \cup P'$ and $P \odot P'$ are the least upper bound and the greatest lower bound, respectively, of the equivalence classes of $(P, c)$ and $(P', c')$ in $\mathcal{P}_k'$.

*Proof.* We first show that (the equivalence class of) $P \cup P'$ is the least upper bound of (the equivalence classes of) $P$ and $P'$. It is clear that both $P$ and $P'$ are homomorphic to $P \cup P'$. Assume that $P$ and $P'$ are homomorphic to a
Then there are homomorphisms \( h : P \to Q, h' : P' \to Q \). Then \( h \cup h' \) is a homomorphism of \( P \cup P' \) to \( Q \). Thus, \( P \cup P' \) is indeed the least upper bound of \( P \) and \( P' \).

Then we show that (the equivalence class of) \( P \otimes P' \) is the greatest lower bound of (the classes of) \( P \) and \( P' \). The first projection \( p_1 : P \otimes P' \to P \), \( p_1(x, y) = x \), is a homomorphism of \( P \otimes P' \) to \( P \). First of all, if \( (x, y) \leq (x', y') \) then \( p_1(x, y) = x \leq x' = p_1(x', y') \); and it is obvious that \( p_1 \) is label-preserving. Similarly, the second projection \( p_2 : P \otimes P' \to P' \), \( p_2(x, y) = y \), is a homomorphism of \( P \otimes P' \) to \( P' \). Assume then that \( (Q, d) \) is homomorphic to both \( P \) and \( P' \). Then there are homomorphisms \( h : Q \to P, h' : Q \to P' \). But then the mapping \( (h, h') : Q \to P \otimes P' \) is a homomorphism. First of all, for every \( x \in Q \), \( c(h(x)) = d(x) = c'(h'(x)) \) so we have that \( (h, h')(x) \in P \otimes P' \) and 
\[
\begin{align*}
(x, y) \in (P, c_1) \otimes (P_2, c_2) \text{ and } (x, y) \in (P_3, c_3) \\
(x, y) \in (P, c_1) \otimes (P_2, c_2) \vee (x, y) \in (P_3, c_3) \\
(x, y) \in (P_1, c_1) \otimes (P, c_2) \vee (x, y) \in (P, c_1) \otimes (P_3, c_3) \\
\end{align*}
\]
Thus, we have in fact the equality
\[
(P_1, c_1) \otimes ((P_2, c_2) \cup (P_3, c_3)) = ((P_1, c_1) \otimes (P_2, c_2)) \cup ((P_1, c_1) \otimes (P_3, c_3)),
\]
and hence the claimed homomorphic equivalence clearly holds. (The other distributive law \((P_1, c_1) \cup ((P_2, c_2) \otimes (P_3, c_3)) \equiv ((P_1, c_1) \cup (P_2, c_2)) \otimes ((P_1, c_1) \cup (P_3, c_3))\) may be a little bit trickier to verify, but, as it happens, it is of course equivalent to the one just proved.)
The empty $k$-poset $(\emptyset, \emptyset)$ is homomorphic to every $k$-poset, but no nonempty $k$-poset is homomorphic to $(\emptyset, \emptyset)$. Thus, the empty $k$-poset is the minimum of $\mathcal{P}_k^\prime$.

It is also easy to see that the atoms of $(\mathcal{P}_k^\prime, \leq)$ are the various singleton $k$-posets. Furthermore, for every nonempty $k$-poset $(P, c)$, there is an atom $(Q, d)$ such that $(Q, d) \leq (P, c)$.

A $k$-chain $a_1 < a_2 < \cdots < a_n$ with labeling $c$ is called alternating, if $c(a_i) \neq c(a_{i+1})$ for every $1 \leq i \leq n - 1$. The alternation number of a $k$-poset $(P, c)$, denoted $\text{Alt}(P, c)$, is the length $n$ of the longest alternating $k$-chain $a_1 < a_2 < \cdots < a_n$ in $(P, c)$.

**Lemma 3.3.** If $(P, c) \leq (P', c')$, then $\text{Alt}(P, c) \leq \text{Alt}(P', c')$.

**Proof.** Let $h : (P, c) \to (P', c')$ be a homomorphism. If $a_1 < a_2 < \cdots < a_n$ is an alternating chain of maximal length in $(P, c)$, then $h(a_1) < h(a_2) < \cdots < h(a_n)$ and $c'(h(a_i)) = c(a_i) \neq c(a_{i+1}) = c'(h(a_{i+1}))$ for every $1 \leq i \leq n - 1$, so there is an alternating chain of length $n$ in $(P', c')$. Thus, $\text{Alt}(P', c') \geq \text{Alt}(P, c)$. \qed

**Corollary 3.4.** Homomorphically equivalent $k$-posets have the same alternation number.

**Corollary 3.5.** Alternating $k$-chains are minimal.

**Proof.** An alternating $k$-chain is not homomorphic to any $k$-poset of smaller cardinality. \qed

**Proposition 3.6.** There are no maximal elements in $(\mathcal{P}_k^\prime, \leq)$.

**Proof.** Let $(P, c)$ be an arbitrary $k$-poset, and let $\text{Alt}(P, c) = n$. Let $a_1 < a_2 < \cdots < a_n$ be an alternating chain in $(P, c)$. Let $(P', c')$ be the $k$-poset that is obtained by adding to $P$ a new element $x$ which covers $a_n$. Define $c'(a) = c(a)$ for $a \in P$, and let $c'(x) = y$ for some $y \neq c(a_n)$. Then $a_1 < a_2 < \cdots < a_n < x$ is an alternating chain of length $n + 1$ in $(P', c')$. It is clear that $(P, c) \leq (P', c')$ and by Lemma 3.3, $(P', c') \not\leq (P, c)$. Thus, $(P, c) < (P', c')$, and hence $(P, c)$ cannot be a maximal element. \qed

For $k \leq l$, every $k$-poset is also an $l$-poset, and it is clear that $\mathcal{P}_k^\prime$ is a subposet of $\mathcal{P}_l^\prime$. In fact, it can be easily verified that $\mathcal{P}_k^\prime$ is a sublattice and an initial segment of $\mathcal{P}_l^\prime$.

The homomorphyce order $\mathcal{L}_2$ of 2-lattices can be described completely in a simple way. Kosub and Wagner [10] showed that every 2-lattice is homomorphically equivalent to its longest alternating chain. For each $n \geq 1$ and $b \in \{0, 1\}$, denote by $\mathcal{C}(n, b)$ the alternating 2-chain of length $n$ whose smallest element has label $b$. Then it is clear that $(\mathcal{L}_2, \leq)$ is represented by the Hasse diagram in Figure 4; representatives of equivalence classes are indicated in the diagram. This poset has width 2, no infinite descending chains, and it is not a lattice.

Denote by $N$ the 2-poset $\{0, 1, 2, 3\}$ with $0 < 1$, $2 < 3$, $2 < 1$, $c(0) = c(3) = 0$, $c(1) = c(2) = 1$. Then $N \not< \mathcal{C}(3, 0)$ but $N$ is not homomorphically equivalent to any 2-lattice. Thus, for $k \geq 2$, $\mathcal{L}_k^\prime$ is not an initial segment of $\mathcal{P}_k^\prime$ (cf. [14, Proposition 1.9]).
Homomorphicity orders are universal

We consider finite sequences of natural numbers, in other words n-tuples \( a = (a_1, \ldots, a_n) \) with \( a_i \in \omega \) for all \( 1 \leq i \leq n \) and \( n \in \omega \). The length of a sequence \( a \) is denoted by \( |a| \). For two sequences \( a \) and \( b \) we write \( a \leq b \) if and only if \( |a| \geq |b| \) and for each \( i \leq |b| \) it holds that \( a_i \geq b_i \). Denote by \( \mathcal{N} \) the set of all finite sequences of natural numbers. It is clear that \( \leq \) is a partial order on \( \mathcal{N} \).

Let \( \mathcal{N}^* \) be the class of all finite subsets of \( \mathcal{N} \). For \( A, B \in \mathcal{N}^* \), we write \( A \leq_{\mathcal{N}^*} B \) if for each \( a \in A \) there exists a \( b \in B \) such that \( a \leq b \).

Denote by \( \mathcal{V} \) the subposet of \( \mathcal{N} \) consisting of all finite binary sequences, i.e., sequences with elements from the two-element set \( \{0, 1\} \). Let \( \mathcal{V}^* \) be the subposet of \( \mathcal{N}^* \) consisting of all finite subsets of \( \mathcal{V} \). It was shown by Hubička and Nešetřil [7, 8], following the ideas presented by Hedrlín [5], that the partially ordered set \( (\mathcal{V}^*, \leq_{\mathcal{V}^*}) \) is universal in the sense that every countable poset can be embedded into it. See reference [7] to find out how to embed an arbitrary countable poset into \( \mathcal{V}^* \).

We will now show that the homomorphism order \( (\mathcal{P}_k', \leq) \) is universal for every \( k \geq 2 \) by exhibiting an embedding of \( \mathcal{V}^* \) into \( \mathcal{P}_2' \).

We represent each finite sequence \( a = (a_1, \ldots, a_n) \in \mathcal{N} \) by a 2-poset \( \mathcal{Q}(a) \).

First of all, the empty sequence () is represented by the 2-poset \( \mathcal{Q}() \) defined by the Hasse diagram in Figure 5. A nonempty sequence \( (a_1, \ldots, a_n) \) is represented by the 2-poset \( \mathcal{Q}(a_1, \ldots, a_n) = (P, c) \), defined as follows (see Figure 6). The set
of elements of \(Q(a_1, \ldots, a_n)\) is \(P = S \times \{0, 1, 2, 3\}\), where \(S\) is the disjoint union

\[
S = C \cup D \cup \bigcup_{i=1}^{n} E_i \cup \bigcup_{i=1}^{n-1} G_i,
\]

where \(C = \{c_0, c_1\}\), \(D = \{d_0, d_1\}\), \(E_i = \{e_{i0}, e_{i1}, \ldots, e_{in_i}\}\), \(G_i = \{g_i\}\). The covering relations are precisely the following:
- for every \(x \in S\), \((x, 0) < (x, 1) < (x, 2) < (x, 3)\),
- for \(1 \leq i \leq n\), \(1 \leq j \leq n_i\), \((e_{ij}, 0) < (e_{i(j-1)}, 2)\),
- for \(1 \leq i \leq n-1\), \((g_i, 0) < (e_{ia_i}, 1), (e_{i+1a_i}, 1) < (g_i, 1)\),
- \((e_0, 2) < (c_1, 3), (c_1, 2) < (c_0, 3), (d_0, 0) < (d_1, 1), (d_1, 0) < (d_0, 1)\),
- \((e_{10}, 0) < (c_0, 1)\) and \((d_0, 0) < (e_{na_n}, 1)\).

The labeling \(c\) is defined as follows: for each \(x \in S\), \(\{x\} \times 4\) is an alternating 2-chain, and \(c(x, 0) = 0\) if and only if \(x \in \{c_0, d_0, g_1, \ldots, g_{n-1}\}\).

In other words, the representation of \((a_1, \ldots, a_n)\) consists of alternating 2-chains of length 4 “glued” together. Each integer \(a_i\) is represented by a block of \(a_i + 1\) chains with the bottom element labeled 1; consecutive blocks are separated by a gap block which is a chain with the smallest element having label 0; in addition, there are special start and end blocks of two chains each.

The interconnections of the various chains are presented in Figure 6.

**Figure 6: The 2-poset \(Q(a_1, \ldots, a_n)\).**

**Proposition 4.1.** For finite sequences \(a\) and \(b\) of natural numbers, \(a \leq b\) if and only if \(Q(a)\) is homomorphic to \(Q(b)\).

**Proof.** Let \(Q(a) = (P, c), Q(b) = (P', c')\), where \(P = S \times 4, P' = S' \times 4\) and so forth, denoting the sets and elements of \(Q(a)\) by non-primed symbols and those of \(Q(b)\) by primed symbols.

Assume that \(h: Q(a) \to Q(b)\) is a homomorphism. If \(b = ()\), then there is nothing to show, because \(a \leq ()\) for every sequence \(a\). If \(a = ()\), then it is easy to verify that the only 2-poset of the form \(Q(b)\) that \(Q(a)\) can be homomorphic to is \(Q()\), and we have that \((a) \leq ()\).

Assume that \(|a| = n > 0, |b| = n' > 0\). The alternating 2-chains of length 4 in \(Q(a)\) are precisely the subposets of the form \(\{x\} \times 4\) for \(x \in S\) (and similarly for \(Q(b)\)). Therefore \(h\) maps each \(\{x\} \times 4\) to \(\{y\} \times 4\) for some \(y \in S'\). Thus, for all \(x \in S, a, a' \in 4\), \(h(x, a) = (y, b)\) and \(h(x, a') = (y', b')\) imply \(a = b, a' = b', y = y'\). So in fact \(h\) has the form \(h(x, a) = (\gamma(x), a)\) for some \(\gamma: S \to S'\), and it suffices to analyze the first component \(\gamma\) of \(h\).
In order for $h$ to preserve the comparabilities within the start block $C \times 4$, it must map the start block $C \times 4$ to the start block $C' \times 4$, so $\gamma(c_i) = c_i'$ for $i = 0, 1$. Similarly, the end block $D \times 4$ must be mapped to the end block $D' \times 4$, and so $\gamma(d_i) = d_i'$ for $i = 0, 1$. Then $\gamma(e_{10}) = e_{10}'$, and it is clear that $\gamma(x) \notin C'$ for $x \notin C$. It also follows that $\gamma[E_i] = E_i'$ and $\gamma(g_i) = g_i'$ for $1 \leq i \leq n'$, and $\gamma(g_i) = d_0'$ and $\gamma[E_i] = \{d_i'\}$ for $i > n'$. This implies that $n \geq n'$ and $a_i \geq b_i$ for $i \leq n'$, i.e., $a \leq b$.

For the converse implication, if $a \leq b$, then it is straightforward to construct a homomorphism of $Q(a)$ to $Q(b)$.

**Remark.** It is clear from the above proof that any homomorphism $h : Q(a) \rightarrow Q(b)$ is surjective. This implies in particular that every endomorphism of $Q(a)$ is surjective and hence injective, $Q(a)$ being finite. By Lemma 2.2, the 2-posets $Q(a)$ are minimal.

**Proposition 4.2.** For $k \geq 2$, $N^*$ can be embedded in $P_k'$.

**Proof.** Define the mapping $\phi : N^* \rightarrow P_k'$ by

$$
\phi(A) = \bigcup_{a \in A} Q(a)
$$

(with a disjoint union of 2-posets). It is straightforward to verify that $\phi$ is an embedding. If $A \leq_{N^*} B$, then for each sequence $a \in A$, there is a $b \in B$ with $a \leq b$, and by Proposition 4.1 there is a homomorphism of $Q(a)$ to $Q(b)$.

Taking the union of these homomorphisms, we obtain a homomorphism of $\phi(A)$ to $\phi(B)$. Conversely, if there is a homomorphism $h : \phi(A) \rightarrow \phi(B)$, then the restriction of $h$ to any connected component $Q(a)$ of $\phi(A)$ is a homomorphism of $Q(a)$ to some connected component $Q(b)$ of $\phi(B)$, and again by Proposition 4.1, $A \leq_{N^*} B$.

**Theorem 4.3.** Every countable poset can be embedded into the homomorphicity order $(P_k', \leq)$ of $k$-posets for $k \geq 2$.

**Proof.** The claim follows from Proposition 4.2 and from the fact that the sub-poset $V^*$ of $N^*$ is universal.

We represented finite sets $A$ of finite integer sequences by 2-posets the connected components of which are the 2-posets $Q(a)$ for $a \in A$. The construction can be slightly modified so that the representations are connected 2-posets. This can be achieved as follows. Instead of forming disjoint unions of the $Q(a)$’s, we let the 2-posets $Q(a)$ have a common start block and leave the remaining parts disjoint. Then the union will be a connected 2-poset with one start block from which spread out the branches representing the various sequences in $A$. In this way we obtain the following strengthening of Theorem 4.3.

**Theorem 4.4.** Every countable poset can be embedded in the homomorphicity order of connected $k$-posets for $k \geq 2$.

We will establish an analogous result for $k$-lattices. As illustrated by Figure 4, the homomorphicity order of 2-lattices has only finite width and is far from being universal. However, with 3-lattices we can construct a suitable representation for $N^*$. 10
The covering relations are precisely the following: $c$, $\lambda$, $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\zeta$, $\eta$, $\theta$. The sets $E$ and $P$ are as follows; see Figure 7. The empty set is represented by the empty 3-lattice $\mathcal{E}(\emptyset) = (\emptyset, \emptyset)$. A nonempty set $A$ of sequences is represented by the 3-lattice $\mathcal{E}(A) = (P, c)$ whose set of elements is the disjoint union

$$P = \{0, 1, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \cup \bigcup_{a \in A} Z(a),$$

where $Z(a) = \{\lambda\}$ for $a = \emptyset$, and for nonempty sequences $a = (a_1, \ldots, a_n)$, $Z(a) = Z'(a) \times \{0, 1\}$, where

$$Z'(a) = \{g_{a_1}, \ldots, g_{a_{n-1}}\} \cup \bigcup_{i=1}^{n} \{f_{a_0}^a, f_{a_1}^a, \ldots, f_{a_n}^a\}.$$

The smallest and the greatest elements of $\mathcal{E}(A)$ are 0 and 1, respectively. The $\alpha_i$’s are on the left and the $\beta_i$’s are on the right in the Hasse diagram in Figure 7. The sets $Z(a)$ are represented by the zigzag figures in the center of the diagram.

The labeling $c$ is defined as follows:
- $c(0) = c(\alpha_2) = c(1) = 2$, $c(\alpha_1) = c(\alpha_3) = c(\beta_1) = c(\beta_3) = 0$, $c(\beta_2) = 1$.
- For every nonempty sequence $a = (a_1, \ldots, a_n) \in A$, $c(f_{a_i}^a, b) = 1$, $c(g_{a_i}^a, b) = 0$ for all $1 \leq i \leq n$, $0 \leq j \leq a_i$ and $b \in \{0, 1\}$.
- If $(\emptyset) \in A$, then $c(\lambda) = 1$.

The covering relations are precisely the following:
- $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1, 0 < \beta_1 < \beta_2 < \beta_3 < 1$.
- For every nonempty sequence $a = (a_1, \ldots, a_n) \in A$, $0 < (x, 0) < (x, 1) < 1$ for all $x \in Z'(a)$; $\alpha_1 < (f_{a_0}^a, 1)$; $f_{a_i}^a, 0) < (f_{a_{i+1}}^a, 1)$ for all $1 \leq i \leq n$, $0 \leq j \leq a_i - 1$; $(f_{a_n}^a, 0) < (g_{a_i}^a, 1)$ and $(g_{a_i}^a, 0) < (f_{a_1}^a, 1)$ for all $1 \leq i \leq n - 1$; $(f_{a_n}^a, 0) < \beta_3$. 

Figure 7: The 3-lattice $\mathcal{E}(A)$. 

We present a finite set $A$ of finite sequences by a 3-lattice $\mathcal{E}(A)$, defined as follows: see Figure 7. The empty set is represented by the empty 3-lattice $\mathcal{E}(\emptyset) = (\emptyset, \emptyset)$. A nonempty set $A$ of sequences is represented by the 3-lattice $\mathcal{E}(A) = (P, c)$ whose set of elements is the disjoint union $P = \{0, 1, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \cup \bigcup_{a \in A} Z(a)$, where

$Z(a) = \{\lambda\}$ for $a = \emptyset$, and for nonempty sequences $a = (a_1, \ldots, a_n)$, $Z(a) = Z'(a) \times \{0, 1\}$, where

$Z'(a) = \{g_{a_1}, \ldots, g_{a_{n-1}}\} \cup \bigcup_{i=1}^{n} \{f_{a_0}^a, f_{a_1}^a, \ldots, f_{a_n}^a\}$.
• If \( \varepsilon \in A \), then \( 0 < \lambda < 1 \), \( \alpha_1 < \lambda < \beta_3 \).

**Proposition 4.5.** \( A \preceq_{\mathcal{N}} B \) if and only if \( \mathcal{E}(A) \) is homomorphic to \( \mathcal{E}(B) \).

**Proof.** It is not difficult to verify this claim, using the same kind of reasoning as in the proof of Propositions 4.1 and 4.2. The only alternating chains of length 5 are \( 0 < \alpha_1 < \alpha_2 < \alpha_3 < 1 \) and \( 0 < \beta_1 < \beta_2 < \beta_3 < 1 \), and any homomorphism must map the \( \alpha_i \)-chain onto the \( \alpha_i \)-chain and the \( \beta_i \)-chain onto the \( \beta_i \)-chain. This ensures that any zigzag \( Z(a) \) is mapped homomorphically to a zigzag \( Z(b) \) for some \( b \in B \) such that \( a \leq b \). If \( |a| > |b| \), then the “overflow” of \( Z(a) \) is absorbed by the chain \( \beta_1 < \beta_2 < \beta_3 \). We leave the details to the reader. \( \square \)

**Theorem 4.6.** Every countable poset can be embedded in the homomorphicity order \( (\mathcal{L}_k, \preceq) \) of \( k \)-lattices for \( k \geq 3 \).

**Proof.** The claim follows from Proposition 4.5 and from the fact that the subposet \( \mathcal{N}^* \) of \( \mathcal{N}^* \) is universal. \( \square \)

## 5 Representation of \( k \)-posets by directed graphs

Let \( (P, c) \) be a \( k \)-poset. We associate with \( (P, c) \) the directed graph \( G(P, c) = (V, E) \), defined as \( V = P \cup S \cup k \cup \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \) (disjoint unions), where \( S = \{(x, y) \in P \times P : x \leq y \in P\} \), and the set \( E \) of edges is defined as follows:

- \( (\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_3, \alpha_1), (\beta_1, \beta_2), (\beta_2, \beta_3), (\beta_3, \beta_1), (\alpha_1, 0), (\alpha_2, 0), (k - 1, \beta_1) \in E \),
- for \( i = 0, \ldots, k - 2, (i, i + 1) \in E \),
- for each \( x \in P, (x, c(x)) \in E \),
- for any \( x, y \in P, (x, (x, y)) \in E \) and \( (x, y), y \in E \) if and only if \( x \leq y \),
- there are no other edges.

Denote by \( \mathcal{G} \) the class of all finite directed graphs. Denote by \( \preceq \) the homomorphicity preorder on \( \mathcal{G} \): \( G \preceq G' \) if and only if there is a graph homomorphism of \( G \) to \( G' \).

**Proposition 5.1.** \( (P, c) \preceq (P', c') \) in \( \mathcal{P}_k \) if and only if \( G(P, c) \preceq G(P', c') \) in \( \mathcal{G} \).

**Proof.** Suppose that \( h : (P, c) \to (P', c') \) is a homomorphism. Define a mapping \( g : G(P, c) \to G(P', c') \) as follows:

\[
g(v) = \begin{cases} 
  v, & \text{if } v \in k \cup \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}, \\
  h(v) & \text{if } v \in P, \\
  (h(x), h(y)), & \text{if } v \in S \text{ and } v = (x, y).
\end{cases}
\]

We shall verify that \( g \) is a homomorphism. Denote by \( E \) and \( E' \) the edge sets of \( G(P, c) \) and \( G(P', c') \), respectively. Let \( (u, v) \in E \). There are four different cases to consider.

Case 1. If \( u, v \in k \cup \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \), then \( (g(u), g(v)) = (u, v) \in E' \).

Case 2. If \( u \in P, v \in k \), then \( g(u) = h(u) \in P', g(v) = v \in k \), and since \( c'(h(u)) = c(u) = v \), we have that \( (g(u), g(v)) = (h(u), v) \in E' \).
Case 3. If \( u \in P, v \in S \), then \( v = (u, u') \) for some \( u \leq u' \) in \( P \). Then \( g(u) = h(u), g(v) = (h(u), h(u')) \), and since \( h \) is a homomorphism, we have that \( h(u) \leq h(u') \) in \( P' \), so \( (h(u), h(u')) \) is indeed a vertex of \( G(P', c') \) and \((g(u), g(v)) = (h(u), (h(u), h(u')))) \( \in E' \).

Case 4. If \( u \in S, v \in P \), then \( u = (v', v) \) for some \( v' \leq v \) in \( P \). Then \( g(v) = h(v'), g(u) = (h(v'), h(v)) \), and since \( h \) is a homomorphism, we have that \( h(v') \leq h(v) \) in \( P' \), so \( (h(v'), h(v)) \) is indeed a vertex of \( G(P', c') \) and \((g(u), g(v)) = ((h(v'), h(v)), h(v)) \in E' \).

In all four cases we have that \((g(u), g(v)) \in E' \), so we conclude that \( g \) is indeed a homomorphism.

Assume then that \( g : G(P, c) \to G(P', c') \) is a graph homomorphism. We show that the restriction of \( g \) to \( P \) is a homomorphism of \((P, c)\) to \((P', c')\). Denote \( G = G(P, c), G' = G(P', c') \).

Since \( \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_1 \) and \( \beta_1 \to \beta_2 \to \beta_3 \to \beta_1 \) are the only 3-cycles in \( G \) and \( G' \), any homomorphism must map each of these 3-cycles to either one of these 3-cycles. We still observe that in these 3-cycles, \( \alpha_1 \) and \( \alpha_2 \) are the only vertices with a common out-neighbour, namely 0, and \( \beta_1 \) and \( \beta_2 \) are the only vertices with a common in-neighbour, namely \( k - 1 \), so we must have that \( g(v) = v \) for \( v \in \{\alpha_1, \alpha_1, \alpha_3, \beta_1, \beta_2, \beta_3\} \). This in turn forces \( g(v) = v \) for any \( v \in k \).

Since \( u \to (u, u) \to u \) for \( u \in P \) are the only 2-cycles (i.e., symmetric edges) in \( G \) and \( v \to (v, v) \to v \) for \( v \in P' \) are the only 2-cycles in \( G' \), we must have that for each \( u \in P, v \in P' \), \( g\{(u, u), u\} = \{v, (v, v)\} \) for some \( v \in P' \). It is not possible that \( g(u) = (v, v) \), because in this case the edge \((u, c(u))\) of \( G \) would not be preserved (there is no edge \((v, v, i)\) for any \( v \in P', i \in k \) in \( G' \)). Thus, \( g(u) = v, g((u, u)) = (v, v) \).

Consider then the edges \((u, (u, u'))\) and \((u, u', u')\) of \( G \) for \( u \leq u' \) in \( P \). We have already observed that \( g(u) = v \) and \( g(u') = v' \) for some \( v, v' \in P' \). It is only possible that \( g((u, u')) = (v, v') \) and \((v, (v, v')) \in E(G') \). Thus \( v \leq v' \), and so \( g \) is order-preserving.

We also observe that for an edge \((u, c(u)) \in P \times k \) of \( G \), we have that \((g(u), g(c(u))) \in (g(u), c(u))) \) an edge of \( G' \). Thus \( c'(g(u)) = c(u) \), so the labels are also preserved.

We conclude that the mapping \( h : (P, c) \to (P', c') \) defined as \( h(x) = g(x) \) is a homomorphism.

Remark. It is obvious from the proof of the above theorem that the category of \( k \)-posets and homomorphisms is isomorphic with the category of digraph representations of \( k \)-posets and graph homomorphisms.

Remark. Sections 4 and 5 actually provide a proof that the homomorphicity order of directed graphs (of odd girth 3) is a universal partial order (which is of course a well-known fact).

6 Concluding remarks

We examined the homomorphicity order \( P'_k \) of finite \( k \)-posets, as well as the homomorphicity order \( L'_k \) of finite \( k \)-lattices. In summary, we showed that \( P'_k \) is a distributive lattice, and for \( k \geq 2 \), \( P'_k \) is universal in the sense that every countable poset can be embedded into it. For \( k \geq 3 \), \( L'_k \) is also universal.
We also represented $k$-posets by directed graphs and established a categorical isomorphism between $k$-posets and their digraph representations.

The present study bears some connections to our earlier work. In [11], we studied so-called $C$-subfunction relations. For a fixed nonempty base set $A$, for a class $C$ of operations on $A$, and for operations $f$ and $g$ on $A$, we say that $f$ is a $C$-subfunction of $g$, denoted $f \leq_C g$, if $f = g(h_1, \ldots, h_n)$ for some $h_1, \ldots, h_n \in C$.

The relation $\leq_C$ is a quasiorder on the set $O_A$ of all operations on $A$ if and only if the defining class $C$ is a clone. Of course, such a quasiorder induces a partial order on the quotient $O_A/\equiv_C$ by the induced equivalence $\equiv_C$.

Assume that $A$ is finite and $|A| \geq 3$, and denote by $M_{\leq}$ the clone of monotone functions with respect to a partial order $\leq$ on $A$. Our analysis of $M_{\leq}$-subfunction relations [11] can be strengthened by Theorem 4.6. Namely, the universality of $(L'_k, \leq)$ for $k \geq 3$ implies that the $M_{\leq}$-subfunction partial orders are also universal whenever $(A, \leq)$ is not an antichain.

References


