ON THE EFFECT OF VARIABLE IDENTIFICATION ON THE
ESSENTIAL ARITY OF FUNCTIONS

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Abstract. We show that every function of several variables on a finite set of
k elements with n > k essential variables has a variable identification minor
with at least n − k essential variables. This is a generalization of a theorem of
Salomaa on the essential variables of Boolean functions. We also strengthen
Salomaa’s theorem by characterizing all the Boolean functions f having a
variable identification minor that has just one essential variable less than f.

1. Introduction

Theory of essential variables of functions has been developed by several authors
[2, 6, 11, 13]. In this paper, we discuss the problem how the number of essential
variables is affected by identification of variables (diagonalization). Salomaa [11]
proved the following two theorems: one deals with operations on arbitrary finite
sets, while the other deals specifically with Boolean functions. We denote the
number of essential variables of f by ess f.

Theorem 1. Let A be a finite set with k elements. For every n ≤ k, there exists
an n-ary operation f on A such that ess f = n and every identification of variables
produces a constant function.

Thus, in general, essential variables can be preserved when variables are identified
only in the case that n > k.

Theorem 2. For every Boolean function f with ess f ≥ 2, there is a function g
obtained from f by identification of variables such that ess g ≥ ess f − 2.

Identification of variables together with permutation of variables and cylindrifi-
cation induces a quasi-order on operations whose relevance has been made apparent
by several authors [3, 7, 8, 9, 10, 12, 14]. In the case of Boolean functions, this
quasi-order was studied in [4] where Theorem 2 was fundamental in deriving certain
bounds on the essential arity of functions.

In this paper, we will generalize Theorem 2 to operations on arbitrary finite
sets in Theorem 3. We will also strengthen Theorem 2 on Boolean functions in
Theorem 6 by determining the Boolean functions f for which there exists a function
g obtained from f by identification of variables such that ess g = ess f − 1.

2. Variable identification minors

Let A and B be arbitrary nonempty sets. A B-valued function of several variables
on A is a mapping f : A^n → B for some positive integer n, called the arity of f.
A-valued functions on \( A \) are called \emph{operations on} \( A \). Operations on \( \{0, 1\} \) are called \emph{Boolean functions}.

We say that the \( i \)-th variable is \emph{essential} in \( f \), or \( f \) depends on \( x_i \), if there are elements \( a_1, \ldots, a_n, b \in A \) such that
\[
f(a_1, \ldots, a_i, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).
\]
The number of essential variables in \( f \) is called the \emph{essential arity} of \( f \), and it is denoted by \( \text{ess } f \). Thus the only functions with essential arity zero are the constant functions.

For an \( n \)-ary function \( f \), we say that an \( m \)-ary function \( g \) is obtained from \( f \) by \emph{simple variable substitution} if there is a mapping \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, m\} \) such that
\[
g(x_1, \ldots, x_m) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]
In the particular case that \( n = m \) and \( \sigma \) is a permutation of \( \{1, \ldots, n\} \), we say that \( g \) is obtained from \( f \) by \emph{permutation of variables}. For indices \( i, j \in \{1, \ldots, n\} \), \( i \neq j \), if \( x_i \) and \( x_j \) are essential in \( f \), then the function \( f_{i \leftarrow j} \) obtained from \( f \) by the simple variable substitution
\[
f_{i \leftarrow j}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n)
\]
is called a \emph{variable identification minor} of \( f \), obtained by identifying \( x_i \) with \( x_j \). Note that \( \text{ess } f_{i \leftarrow j} < \text{ess } f \), because \( x_i \) is not essential in \( f_{i \leftarrow j} \) even though it is essential in \( f \).

We define a quasiorder on the set of all \( B \)-valued functions of several variables on \( A \) as follows: \( f \leq g \) if and only if \( f \) is obtained from \( g \) by simple variable substitution. If \( f \leq g \) and \( g \leq f \), we denote \( f \equiv g \). If \( f \leq g \) but \( g \nleq f \), we denote \( f < g \). It can be easily observed that if \( f \leq g \) then \( \text{ess } f \leq \text{ess } g \), with equality if and only if \( f \equiv g \).

For a \( B \)-valued function \( f \) of several variables on \( A \), we denote the maximum essential arity of a variable identification minor of \( f \) by
\[
\text{ess}^< f = \max_{g < f} \text{ess } g,
\]
and we define the \emph{arity gap} of \( f \) by \( \text{gap } f = \text{ess } f - \text{ess}^< f \).

### 3. Generalization of Theorem \ref{thm:generalization}

**Theorem 3.** Let \( A \) be a finite set of \( k \geq 2 \) elements, and let \( B \) be a set with at least two elements. Every \( B \)-valued function of several variables on \( A \) with \( n > k \) essential variables has a variable identification minor with at least \( n - k \) essential variables.

In the proof of Theorem \ref{thm:generalization} we will make use of the following theorem due to Salomaa \cite{Salomaa} Theorem 1], which is a strengthening of Yablonski’s \cite{Yablonski} “fundamental lemma”.

**Theorem 4.** Let the function \( f : M_1 \times \cdots \times M_n \to N \) depend essentially on all of its \( n \) variables, \( n \geq 2 \). Then there is an index \( j \) and an element \( c \in M_j \) such that the function
\[
f(x_1, \ldots, x_{j-1}, c, x_{j+1}, \ldots, x_n)
\]
depends essentially on all of its \( n - 1 \) variables.

We also need the following auxiliary lemma.
Lemma 5. Let $f$ be an $n$-ary function with $\text{ess } f = n > k$. Then there are indices $1 \leq i < j \leq k + 1$ such that at least one of the variables $x_1, \ldots, x_{k+1}$ is essential in $f_{i,j}$.

Proof. Since $x_1$ is essential in $f$, there are elements $a_1, \ldots, a_n, b \in A$ such that $f(a_1, a_2, \ldots, a_n) \neq f(b, a_2, \ldots, a_n)$.

Thus there are indices $1 \leq i < j \leq k + 1$ such that $a_i = a_j$. If $i \neq 1$, then it is clear that $x_1$ is essential in $f_{i,j}$. If there are no such $i$ and $j$ with $i \neq 1$, then $i = 1 < j$ and we have that $b = a_l$ for some $1 < l \leq k+1, l \neq j$. For $m = 1, \ldots, n$, let $c_m = a_m$ if $m \notin \{1, j, l\}$ and let $c_m = a_1$ if $m \in \{1, j, l\}$. Then $f(c_1, c_2, \ldots, c_n)$ is distinct from at least one of $f(a_1, a_2, \ldots, a_n)$ and $f(b, a_2, \ldots, a_n)$. If $f(c_1, c_2, \ldots, c_n) \neq f(a_1, a_2, \ldots, a_n)$, then $x_i$ is essential in $f_{i,j}$. If $f(c_1, c_2, \ldots, c_n) \neq f(b, a_2, \ldots, a_n)$, then $x_i$ is essential in $f_{i,j}$. \hfill \Box

Proof of Theorem 3. By Theorem 1 there exist $k + 1$ constants $c_1, \ldots, c_{k+1} \in A$ such that, after a suitable permutation of variables, the function $f(c_1, \ldots, c_{k+1}, x_{k+2}, \ldots, x_n)$ depends on all of its $n - k - 1$ variables. There are indices $1 \leq i < j \leq k + 1$ such that $c_i = c_j$, and by Lemma 5 there are indices $1 \leq l < m \leq k + 1$ such that at least one of the variables $x_1, \ldots, x_{k+1}$ is essential in $f_{l,m}$. With a suitable permutation of variables, we may assume that $i = 1, j = 2, 1 \leq l \leq 3, m = l + 1$.

If one of the variables $x_1, \ldots, x_{k+1}$ is essential in $f_{1,2}$, then we are done. Otherwise we have that for all $a_1, \ldots, a_n \in A$,

$$f(c_1, c_1, c_3, c_4, \ldots, c_{k+1}, a_{k+2}, \ldots, a_n) = f(c_3, c_3, c_4, \ldots, c_{k+1}, a_{k+2}, \ldots, a_n).$$

Thus the variables $x_{k+2}, \ldots, x_n$ are essential in $f_{2,3}$. If one of the variables $x_1, \ldots, x_{k+1}$ is essential in $f_{2,3}$, then we are done. Otherwise we have that for all $a_1, \ldots, a_n \in A$,

$$f(c_3, c_3, c_4, \ldots, c_{k+1}, a_{k+2}, \ldots, a_n) = f(c_3, c_4, c_4, \ldots, c_{k+1}, a_{k+2}, \ldots, a_n),$$

and so the variables $x_{k+2}, \ldots, x_n$ are essential in $f_{3,4}$ and also at least one of $x_1, \ldots, x_{k+1}$ is essential in $f_{3,4}$. \hfill \Box

We would like to remark that our proof is considerably simpler than Salomaa’s original proof of Theorem 2.

4. Strengthening of Theorem 2

It is well-known that every Boolean function is represented by a unique multilinear polynomial over the two-element field. Such a representation is called the Zhegalkin polynomial of $f$. It is clear that a variable is essential in $f$ if and only if it occurs in the Zhegalkin polynomial of $f$. We denote by $\deg p$ the degree of polynomial $p$. If $p$ is the Zhegalkin polynomial of $f$, then we denote the Zhegalkin polynomial of $f_{i,j}$ by $p_{i,j}$. Note that the only polynomials of degree 0 are the constant polynomials.

Theorem 6. Let $f$ be a Boolean function with at least 2 essential variables. Then the arity gap of $f$ is 2 if and only if the Zhegalkin polynomial of $f$ is of one of the following special forms:

- $x_{i_1} + x_{i_2} + \cdots + x_{i_n} + c$,
\[ x_i x_j + x_i + c, \]
\[ x_i x_j + x_i x_k + x_j x_k + c, \]
\[ x_i x_j + x_i x_k + x_j x_k + x_i + x_j + c, \]

where \( c \in \{0, 1\} \). Otherwise the arity gap of \( f \) is 1.

We prove first an auxiliary lemma that takes care of the functions of essential arity at least 4 whose Zhegalkin polynomial has degree 2.

**Lemma 7.** If \( f \) is a Boolean function with at least four essential variables and the Zhegalkin polynomial of \( f \) has degree two, then the arity gap of \( f \) is one.

**Proof.** Denote the Zhegalkin polynomial of \( f \) by \( p \). We need to consider several cases and subcases.

**Case 1.** Assume first that \( p \) is of the form

\[ p = x_i x_j + x_i x_k + x_j x_k + x_i a_i + x_j a_j + x_k a_k + a, \]

where \( a_i, a_j, a_k \) are polynomials of degree at most 1 and \( a \) is a polynomial of degree at most 2 such that there are no occurrences of variables \( x_i, x_j, x_k \) in \( a_i, a_j, a_k, a \).

**Subcase 1.1.** Assume that \( \deg a_i = \deg a_j = \deg a_k = 0 \). Then \( a \) contains a variable \( x_l \) distinct from \( x_i, x_j, x_k \), and we can write \( a = x_l a' + a'' \), where \( a' \) and \( a'' \) do not contain \( x_l \). Then \( f_{l \rightarrow i} \) is represented by the polynomial

\[ p_{l \rightarrow i} = x_i x_j + x_i x_k + x_j x_k + x_i a' + a'', \]

where all essential variables of \( f \) except for \( x_l \) occur, because no terms cancel, and hence \( \text{gap } f = 1 \).

**Subcase 1.2.** Assume that at least one of \( a_i, a_j, a_k \) has degree 1, say \( \deg a_i = 1 \). Then \( a_i \) contains a variable \( x_l \) distinct from \( x_i, x_j, x_k \), and so \( a_i = x_l + a'_i \), where \( a'_i \) has degree at most 1 and does not contain \( x_l \). Consider

\[ p_{j \rightarrow k} = x_k (1 + a_j + a_k) + x_l a_i + a. \]

If all essential variables of \( f \) except for \( x_j \) occur in \( p_{j \rightarrow k} \), then \( \text{gap } f = 1 \) and we are done. Otherwise we need to analyze three different subcases.

**Subcase 1.2.1.** Assume that variable \( x_k \) occurs in \( p_{j \rightarrow k} \) but there is a variable \( x_l \) that occurs in \( a_i \) and \( a_k \) but not in \( a_i \) nor in \( a \) such that \( x_l \) does not occur in \( p_{j \rightarrow k} \) (due to some cancelling terms in \( a_j \) and \( a_k \)). Write \( a_j = x_l + a'_j, a_k = x_l + a'_k \), and consider

\[ p_{j \rightarrow l} = x_i x_l + x_i x_k + x_l x_k + x_i a_i + x_l + x_k a'_l + x_k x_l + x_k a'_k + a \]

\[ = x_i x_l + x_i x_k + x_l a_i + x_l + x_l a'_l + x_k a'_k + a. \]

Every essential variable of \( f \) except for \( x_j \) occurs in \( p_{j \rightarrow l} \), and hence \( \text{gap } f = 1 \).

**Subcase 1.2.2.** Assume that \( x_k \) does not occur in \( p_{j \rightarrow k} \). In this case \( a_j = a_k + 1 \). Consider

\[ p_{j \rightarrow k} = x_i (1 + a_i + a_j) + x_k a_k + a. \]

If any term of \( a_j \) is cancelled by a term of \( a_i \), it still remains as a term of \( a_k \), and hence all variables occurring in \( a_i, a_j, a_k \) occur in \( p_{j \rightarrow k} \). If both \( x_i \) and \( x_k \) also occur in \( p_{j \rightarrow k} \), then all essential variables of \( f \) except for \( x_j \) occur in \( p_{j \rightarrow k} \), and so \( \text{gap } f = 1 \).

If \( x_k \) does not occur in \( p_{j \rightarrow k} \), then \( a_k = 0 \) and so \( a_j = 1 \). Then

\[ p_{l \rightarrow k} = x_i x_j + x_i x_k + x_j x_k + x_i + x_i a'_i + x_j + a, \]

and every essential variable of \( f \) except for \( x_j \) occurs in \( p_{l \rightarrow k} \). Thus \( \text{gap } f = 1 \).
If \( x_i \) does not occur in \( p_{j-i} \), then \( a_j = a_i + 1 \), and hence \( a_i = a_k \). Consider then
\[
p_{i-k} = x_k(1 + a_i + a_k) + x_ja_j + a = x_k + x_ja_j + a.
\]
Again all essential variables of \( f \) except for \( x_i \) occur in \( p_{i-k} \), and so gap \( f = 1 \).

**Subcase 1.2.3.** Assume that both \( x_i \) and \( x_k \) occur in \( p_{j-k} \) but there is a variable \( x_l \) occurring in \( a_i \) and in \( a_j \) but not in \( a_k \) nor in \( a \) such that \( x_k \) does not occur in \( p_{j-k} \) (due to some cancelling terms in \( a_i \) and \( a_j \)). Write \( a_i = x_l + a_i', a_j = x_l + a_j' \), and consider
\[
p_{j-l} = x_i x_l + x_i x_k + x_l x_k + x_i x_l + x_i a_i' + x_l + x_i a_i' + x_k a_k + a
\]
\[
= x_i x_k + x_i x_k + x_i a_i' + x_l + x_i a_i' + x_k a_k + a.
\]
Every essential variable of \( f \) except for \( x_j \) occurs in \( p_{j-l} \), and so gap \( f = 1 \).

**Case 2.** Assume then that \( p \) is of the form
\[
p = x_i x_j + x_i x_k a_k + x_i a_i + x_j a_j + x_k a_k + a,
\]
where \( a_k \) is a polynomial of degree 0; \( a_i \), \( a_j \), \( a_k \) are polynomials of degree at most 1; and \( a \) is a polynomial of degree at most 2 such that variables \( x_i \), \( x_j \), \( x_k \) do not occur in \( a_k \), \( a_i \), \( a_j \), \( a_k \), \( a \). Note that \( a_{ik} \) and \( a_k \) cannot both be 0, for otherwise \( x_k \) would not occur in \( p \). Consider
\[
p_{j-l} = x_i(1 + a_i + a_j) + x_i x_k a_k + x_k a_k + a.
\]
By the above observation that \( a_{ik} \) and \( a_k \) are not both 0, \( x_k \) occurs in \( p_{j-i} \). If all essential variables of \( f \) except for \( x_j \) occur in \( p_{j-i} \), then gap \( f = 1 \) and we are done. Otherwise we distinguish between two cases.

**Subcase 2.1.** Assume that \( x_i \) does not occur in \( p_{j-i} \). In this case \( a_j = a_i + 1 \), \( a_{ik} = 0 \), and \( a_k \neq 0 \). Consider
\[
p_{i-k} = x_j x_k + x_k a_k + x_k a_i + x_j a_j + x_k a_k + a
\]
\[
= x_j x_k + x_k (a_i + a_k) + x_j + x_j a_i + a.
\]
Both \( x_j \) and \( x_k \) occur in \( p_{i-k} \), because the term \( x_j x_k \) cannot be cancelled. If any term of \( a_i \) is cancelled by a term of \( a_k \), it still remains in \( x_j a_i \). Thus, all essential variables of \( f \) except for \( x_i \) occur in \( p_{i-k} \), and hence gap \( f = 1 \).

**Subcase 2.2.** Assume that \( x_i \) occurs in \( p_{j-i} \) but there is a variable \( x_l \) occurring in \( a_i \) and \( a_j \) but not in \( a_{ik} \), \( a_k \), nor in \( a \) such that \( x_l \) does not occur in \( p_{j-i} \) (due to some cancelling terms in \( a_i \) and \( a_j \)). Consider
\[
p_{k-l} = x_i x_j + x_i x_l a_{ik} + x_i a_i + x_j a_j + x_i a_{ik} + a.
\]
If \( a_{ik} = 1 \), then the terms \( x_i x_l \) in \( x_i a_i \) and in \( x_i x_l a_{ik} \) cancel each other. These are the only terms that may be cancelled out. Nevertheless, \( x_l \) occurs also in \( a_j \), and so all essential variables of \( f \) except for \( x_k \) occur in \( p_{k-l} \). Therefore gap \( f = 1 \) also in this case.

**Proof of Theorem:** Denote the Zhegalkin polynomial of \( f \) by \( p \). It is straightforward to verify that if \( p \) has one of the special forms listed in the statement of the theorem, then \( f \) does not have a variable identification minor of essential arity \( ess f - 1 \) but it has one of essential arity \( ess f - 2 \). For the converse implication, we will prove by induction on \( ess f \) that if \( p \) is not of any of the special forms, then there is a variable identification minor \( g \) of \( f \) such that \( ess g = ess f - 1 \), i.e., \( f \) has arity gap 1.
If \( \text{ess } f = 2 \) and \( p \) is not of any of the special forms, then \( p = x_i x_j + c \) or \( p = x_i x_j + x_i + x_j + c \) where \( c \in \{0,1\} \), and in both cases \( p_{j-i} = x_i + c \). In this case \( \text{gap } f = 1 \).

If \( \text{ess } f = 3 \), then \( p \) has one of the following forms

\[
\begin{align*}
x_i x_j x_k + x_i x_j + x_i x_k + x_j x_k + a_i x_i + a_j x_j + a_k x_k + c, \\
x_i x_j x_k + x_i x_k + x_j x_k + a_i x_i + a_j x_j + a_k x_k + c, \\
x_i x_j x_k + x_i x_j + a_i x_i + a_j x_j + a_k x_k + c, \\
x_i x_j + x_i x_k + x_j x_k + x_i + x_j + x_k + c, \\
x_i x_j + x_i x_k + x_j x_k + x_i + x_j + c, \\
x_i x_j + x_i x_k + a_i x_i + a_j x_j + a_k x_k + c, \\
x_i x_k + a_i x_i + a_j x_j + a_k x_k + c,
\end{align*}
\]

where \( a_i, a_j, a_k, c \in \{0,1\} \). It is easy to verify that in each case \( p_{j-i} \) contains the term \( x_i x_k \), and hence both \( x_i \) and \( x_k \) are essential in \( f_{j-i} \), and so \( \text{gap } f = 1 \).

For the sake of induction, assume then that the claim holds for \( 2 \leq \text{ess } f < n \), \( n \geq 4 \). Consider the case that \( \text{ess } f = n \). Since the case where \( \deg p = 1 \) is ruled out by the assumption that \( p \) does not have any of the special forms and the case where \( \deg p = 2 \) is settled by Lemma 7, we can assume that \( \deg p \geq 3 \). Choose a variable \( x_m \) from a term of the highest possible degree in \( p \), and write

\[ p = x_m q + r, \]

where the polynomials \( q \) and \( r \) do not contain \( x_k \). We clearly have that \( \deg q = \deg p - 1 \), and \( q \) and \( r \) represent functions with less than \( n \) essential variables. Of course, every essential variable of \( f \) except for \( x_m \) occurs in \( q \) or \( r \). We have three different cases to consider, depending on the comparability under inclusion of the sets of variables occurring in \( q \) and \( r \).

**Case 1.** Assume that there is a variable \( x_i \) that occurs in \( q \) but does not occur in \( r \), and there is a variable \( x_j \) that occurs in \( r \) but does not occur in \( q \). Write

\[ q = x_i q' + q'', \quad r = x_j t' + t'', \]

where \( q', q'', t', t'' \) do not contain \( x_i, x_j \). Then

\[ p = x_m x_i q' + x_m q'' + x_j t' + t'', \]

and we have that

\[ p_{j-i} = x_m x_i q' + x_m q'' + x_i t' + t'', \]

where no terms can cancel. Hence all essential variables of \( f \) except for \( x_j \) are essential in \( f_{j-i} \) and so \( \text{gap } f = 1 \).

**Case 2.** Assume that every variable occurring in \( r \) occurs in \( q \). In this case \( q \) represents a function \( q \) of essential arity \( \text{ess } f - 1 \), containing all essential variables of \( f \) except for \( x_m \). We also have that \( \deg q = \deg p - 1 \geq 2 \).

**Subcase 2.1.** If \( \text{ess } f \geq 5 \), then \( \text{ess } q \geq 4 \), and we can apply the inductive hypothesis, which tells us that there are variables \( x_i \) and \( x_j \) such that \( \text{ess } q_{i-j} = \text{ess } q - 1 \). Hence \( f_{i-j} \) is represented by the polynomial \( p_{i-j} = x_m q_{i-j} + r_{i-j} \), and all essential variables of \( f \) except for \( x_i \) occur in \( p_{i-j} \), since no terms can cancel between \( x_m q_{i-j} \) and \( r_{i-j} \). Thus \( \text{gap } f = 1 \).

**Subcase 2.2.** If \( \text{ess } f = 4 \), then \( \text{ess } q = 3 \), and we can apply the inductive hypothesis as above unless \( q = x_i x_j + x_i x_k + x_j x_k + c \) or \( q = x_i x_j + x_i x_k + x_j x_k + c \)
If this is the case, consider first the case where \( q \) contains a variable \( x_l \in \{x_i, x_j, x_k\} \) that does not occur in \( r \). Consider then

\[
p_{m-l} = x_l q + r.
\]

Then \( x_l q \) contains the term \( x_i x_j x_k \), which cannot be cancelled. Namely, all other terms of \( x_l q \) have degree at most 2, and since there are at most two variables occurring in \( r \), the terms of \( r \) also have degree at most 2. Thus, all variables of \( f \) except for \( x_m \) occur in \( p_{m-l} \), and so the arity gap of \( f \) is 1.

Consider then the case that \( q \) and \( r \) contain the same variables, i.e., \( x_i, x_j, x_k \). If \( \deg r \leq 2 \), then it is easily seen that \( p_{m-l} \) contains the term \( x_i x_j x_k \), and all essential variables of \( f \) except for \( x_m \) are essential in \( f_{m-l} \). Otherwise, we can apply the inductive hypothesis on the function \( r \) represented by \( r \) and we obtain variables \( x_a \) and \( x_b \) such that \( \text{ess} r_{a-b} = \text{ess} r - 1 \). It can be easily verified that no identification of variables brings \( q \) into the zero polynomial, so \( x_m \) and two other variables will occur in \( p_{\alpha-\beta} = x_m q_{\alpha-\beta} + r_{\alpha-\beta} \). We have that gap \( f = 1 \) also in this case.

Case 3. Assume that every variable occurring in \( q \) occurs in \( r \) but there is a variable \( x_l \) that occurs in \( r \) but does not occur in \( q \). If \( \deg r = 1 \), then \( r = x_l + r' \) where \( r' \) does not contain \( x_l \). Then \( p_{m-l} = x_l q + x_l + r' \), where the only term that may cancel out is \( x_l \), and this happens if \( q \) has a constant term 1. Nevertheless, \( x_l \) occurs in \( t_{m-l} \) because deg \( q \geq 2 \). Of course, all other essential variables of \( f \) except for \( x_m \) also occur in \( p_{m-l} \), so gap \( f = 1 \). We may thus assume that \( \deg r \geq 2 \).

Subcase 3.1. Assume first that ess \( f = 4 \) (in which case \( r \) contains three variables and \( q \) contains at most two variables) and \( r = x_i x_j + x_i x_k + x_j x_k + x_i + x_j + c \) or \( r = x_i x_j + x_i x_k + x_j x_k + x_i + x_j + c \). Since we assume that \( \deg p \geq 3 \), we have that \( \deg q \geq 2 \) and hence \( q \) contains at least two variables. Thus exactly two variables occur in \( q \) and so also \( \deg q = 2 \). Then \( q = x_\alpha x_\beta + b_1 x_\alpha + b_2 x_\beta + d \) where \( \alpha, \beta \in \{i, j, k\} \) and \( b_1, b_2, d \in \{0, 1\} \). Let \( \gamma \in \{i, j, k\} \setminus \{\alpha, \beta\} \). Then \( p_{m-\gamma} \) contains the term \( x_i x_j x_k \), and hence all essential variables of \( f \) except for \( x_m \) occur in \( p_{m-\gamma} \), and so gap \( f = 1 \).

Subcase 3.2. Assume then that ess \( f > 4 \) or ess \( f = 4 \) but \( r \) does not have any of the special forms. In this case we can apply the inductive hypothesis on the function \( r \) represented by \( r \). Let \( x_i \) and \( x_j \) be such that ess \( r_{i-j} = ess r - 1 \). If \( q_{j-i} \neq 0 \), then \( x_m \) and all other essential variables of \( f \) except for \( x_j \) occur in \( p_{j-i} \), and we are done—the arity gap of \( f \) is 1. We may thus assume that \( q_{j-i} = 0 \).

Write \( q \) and \( r \) in the form

\[
q = x_i x_j a_1 + x_i a_2 + x_j a_3 + a_4,
\]
\[
r = x_i x_j b_1 + x_i b_2 + x_j b_3 + b_4,
\]

where the polynomials \( a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \) do not contain \( x_i, x_j \). Define the polynomials \( q_1, \ldots, q_7 \) as follows (cf. the proof of Theorem 4 in Salomaa [11]):

- \( q_1 \) consists of the terms common to \( a_1, a_2, \) and \( a_3 \).
- \( q_i, i = 2, 3, \) consists of those terms common to \( a_i \) and \( a_j \) which are not in \( q_1 \).
- \( q_{4+i}, i = 1, 2, 3, \) consists of the remaining terms in \( a_i \).

Define the polynomials and \( r_1, \ldots, r_7 \) similarly in terms of the \( b_j \)'s. Note that for any \( i \neq j \), \( q_i \) and \( q_j \) do not have any terms in common, and similarly \( r_i \) and \( r_j \).
do not have any terms in common. Hence,

\[ q = x_1 x_j (q_1 + q_2 + q_3 + q_5) + x_i (q_1 + q_2 + q_4 + q_6) + x_j (q_1 + q_3 + q_4 + q_7) + a_4, \]

\[ r = x_1 x_j (r_1 + r_2 + r_3 + r_5) + x_i (r_1 + r_2 + r_4 + r_6) + x_j (r_1 + r_3 + r_4 + r_7) + b_4. \]

Identification of \( x_i \) with \( x_j \) yields

\[ q_{j-i} = x_i (q_1 + q_5 + q_6 + q_7) + a_4, \]

\[ r_{j-i} = x_i (r_1 + r_5 + r_6 + r_7) + b_4. \]

Since we are assuming that \( q_{j-i} = 0 \), we have that \( q_1 = q_5 = q_6 = q_7 = a_4 = 0 \).

On the other hand, \( q \neq 0 \), so \( q_2, q_3, q_4 \) are not all zero. Thus

\[ q = x_1 x_j (q_2 + q_3) + x_i (q_2 + q_4) + x_j (q_3 + q_4). \]

All essential variables of \( f \) except for \( x_j \) are contained in \( r_{j-i} \).

Subcase 3.2.1. Assume that there is a variable \( x_i \) occurring in \( b_4 \) that does not occur in \( r_1, r_5, r_6, r_7 \). Consider

\[ p_{m-i} = x_i q + r = x_i q + x_i x_j b_1 + x_i b_2 + x_j b_3 + b_4. \]

Cancelling may only happen between a term of \( x_i q \) and a term of \( r \). No term of \( b_4 \) can be cancelled, because every term of \( x_i q \) contains \( x_i \) or \( x_j \) but the terms of \( b_4 \) do not contain either. The variables that do not occur in \( b_4 \) occur in some terms of \( b_1, b_2, b_3 \) that do not contain \( x_i \). Thus, all essential variables of \( f \) except for \( x_m \) occur in \( p_{m-i} \), and so in this case \( f \) has arity gap 1.

Subcase 3.2.2. Assume that all variables of \( r \) except for \( x_j \) occur already in \( r_1 + r_5 + r_6 + r_7 \). Consider

\[ p_{m-i} = x_1 x_j (q_2 + q_4 + r_1 + r_2 + r_3 + r_5) + \]

\[ x_i (q_2 + q_4 + r_1 + r_2 + r_4 + r_6) + \]

\[ x_j (r_1 + r_3 + r_4 + r_7) + b_4. \]  \hspace{1cm} (1)

Subcase 3.2.2.1. Assume first that \( x_i \) does not occur in \( p_{m-i} \) in (1). Then

\[ q_2 + q_4 + r_1 + r_2 + r_3 + r_5 = 0, \]

\[ q_2 + q_4 + r_1 + r_2 + r_4 + r_6 = 0, \]

and since the \( r_i \)'s do not have terms in common, we have that

\[ r_1 + r_2 = q_2 + q_4, \quad r_3 = r_4 = r_5 = r_6 = 0. \]

Then all variables of \( r \) except for \( x_i, x_j \) occur already in \( r_1 + r_7 \). Consider

\[ p_{m-j} = x_1 x_j (q_3 + q_4 + r_1 + r_2 + r_3 + r_5) + \]

\[ x_i (r_1 + r_2 + r_4 + r_6) + \]

\[ x_j (q_3 + q_4 + r_1 + r_3 + r_4 + r_7) + b_4 \]

\[ = x_i x_j (q_2 + q_3) + \]

\[ x_i (r_1 + r_2) + \]

\[ x_j (q_2 + q_3 + r_2 + r_7) + b_4. \]  \hspace{1cm} (2)

All variables of \( r_1 \) are there on the fifth line of (2). If a term of \( r_7 \) is cancelled by a term of \( q_2 + q_3 \) on the sixth line, it still remains on the fourth line, so all variables of \( r_7 \) are also there. We still need to verify that the variables \( x_i \) and \( x_j \)
are not cancelled out from (2). If \( q_2 + q_3 \neq 0 \) then we are done. Assume then that \( q_2 + q_3 = 0 \), in which case \( q_4 \neq 0 \). Since

\[
q_3 + q_4 + r_1 + r_3 + r_4 + r_7 = q_4 + r_1 + r_7
\]

and \( r_1 + r_7 \) contain all variables of \( r \) except for \( x_i, x_j \), but \( q_4 \) does not, \( q_4 + r_1 + r_7 \neq 0 \), so we also have \( x_j \) in (3). Since

\[
t_1 + t_2 + t_6 = t_1 + t_2 = q_2 + q_4 = q_4 \neq 0,
\]

we have \( x_i \) in (2). Since

\[
t_3 + q_4 + r_1 + r_3 + r_4 + r_7 = q_4 + r_1 + r_7
\]

terms of \( f \) may be cancelled out by terms of \( q_2 + q_4 \) on the first line of (1) but such terms will remain on the second line. Thus the variables of \( r_5 \) occur in \( p_{m-i} \). A similar argument shows that the variables of \( r_6 \) also occur in \( p_{m-i} \). In order for \( f \) to have arity gap 1, we still need to verify that \( x_j \) occurs in \( p_{m-i} \). If \( q_2 + q_4 + r_1 + r_2 + r_3 + r_5 \neq 0 \), then we are done. We may thus assume that

\[
q_2 + q_4 + r_1 + r_2 + r_3 + r_5 = 0.
\]

By the assumption that \( x_j \) occurs in \( p_{m-i} \), the second line of (1) does not vanish, i.e.,

\[
0 \neq q_2 + q_4 + r_1 + r_2 + r_4 + r_6 = r_3 + r_4 + r_5 + r_6.
\]

If the third line of (1) does not vanish either, i.e., \( r_1 + r_3 + r_4 + r_7 \neq 0 \), then we have both \( x_i \) and \( x_j \) and we are done. We may thus assume that \( r_1 + r_3 + r_4 + r_7 = 0 \), i.e., \( r_1 = r_3 = r_4 = r_7 = 0 \). Then all variables of \( r \) except for \( x_i, x_j \) occur already in \( r_5 + r_6 \). Equation (3) implies that \( r_2 + r_5 = q_2 + q_4 \). Consider

\[
p_{m-j} = x_i x_j (q_3 + q_4 + r_1 + r_2 + r_3 + r_5) + \\
\quad x_i (t_1 + t_2 + r_4 + r_6) + \\
\quad x_j (q_5 + q_4 + r_1 + r_3 + r_4 + r_7) + b_4
\]

\[
= x_i x_j (q_2 + q_4) + \\
\quad x_i (q_2 + q_4 + r_5 + r_6) + \\
\quad x_j (q_3 + q_4) + b_4.
\]

Assume first that \( q_2 + q_3 = 0 \), in which case \( q_4 \neq 0 \). If a term of \( r_5 + r_6 \) is cancelled by a term of \( q_4 \) on the second line of (1), it will still remain on the third line. Therefore we have in \( p_{m-j} \) all variables of \( r \) except for \( x_i, x_j \). Since \( r_5 + r_6 \) contains all variables of \( r \) except for \( x_i, x_j \) but \( q_2 + q_4 = q_4 \) does not, the second line of (1) does not vanish, and so we have \( x_i \). We also have \( x_j \) because \( q_3 + q_4 = q_4 \neq 0 \) on the third line. In this case \( f \) has arity gap 1.

Assume then that \( q_2 + q_3 \neq 0 \). Then the first line of (1) does not vanish and both \( x_i \) and \( x_j \) occur in \( p_{m-j} \). If any term of \( r_5 + r_6 \) is cancelled by a term of \( q_2 \) on the second line of (1), it still remains on the first line, and if it is cancelled by a term of \( q_4 \), it remains on the third line. Thus all variables of \( r \) occur in \( p_{m-j} \), and \( f \) has arity gap 1 again. This completes the proof of Theorem 6. \( \square \)

5. Concluding remarks

We do not know whether the upper bound on arity gap given by Theorem 3 is sharp. For base sets \( A \) with \( k \geq 3 \) elements, we do not know whether there exists an operation \( f \) on \( A \) with \( \text{ess} f \geq k + 1 \) and \( \text{gap} f \geq 3 \). We know that for all \( k \geq 2 \),
there are operations on a $k$-element set $A$ with arity gap 2. Consider for instance the quasi-linear functions of Burle [1]. A function $f$ is quasi-linear if it has the form

$$f = g(h_1(x_1) \oplus h_2(x_2) \oplus \cdots \oplus h_n(x_n)),$$

where $h_1, \ldots, h_n : A \to \{0, 1\}$, $g : \{0, 1\} \to A$ are arbitrary mappings and $\oplus$ denotes addition modulo 2. It is easy to verify that if those $h_i$’s that are nonconstant coincide (and $g$ is not a constant map), then $f$ has arity gap 2.

In general, if there is an operation $f$ on a $k$-element set $A$ with with gap $f = m$, then there are operations of arity gap $m$ on all sets $B$ of at least $k$ elements. Namely, it is easy to see that any operation $g$ on $B$ of the form

$$g = \phi(f(\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_n))),$$

where $\gamma : B \to A$ is surjective and $\phi : A \to B$ is injective, satisfies ess $g = \text{ess } f$ and gap $g = \text{gap } f$.

References